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Solvable quantum mechanical examples of broken supersymmetry

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We construct several analytically solvable examples of broken supersymmetry in non-relativistic quantum mechanics. Our examples are motivated from known shape-invariant potentials obeying unbroken supersymmetry and are obtained from them via suitable mapping procedures.

There are a number of exactly solvable problems in non-relativistic quantum mechanics for which the energy eigenvalues and eigenfunctions can be obtained by the method of factorization [1]. The underlying features which permit this procedure to work are unbroken supersymmetry (SUSY) [2–4] and the property of shape-invariance [5]. In supersymmetric quantum mechanics (SUSYQM) one defines a pair of potentials (in units of $2m=\hbar=1$)

$$V_{\pm}(x)=W^2(x)\pm W'(x)\,, \tag{1}$$

where $W(x)$ is known as the superpotential. For unbroken SUSY, the potential V_- has a normalizable ground state wavefunction

$$\psi_0^{(-)}(x)=\exp\left(-\int^x W(x')\,\mathrm{d}x'\right), \tag{2}$$

which corresponds to $E_0^{(-)}=0$. Except for the ground state, the potential V_- has all its energy eigenstates degenerate with those of the partner potential V_+, i.e.

$$E_{n+1}^{(-)}=E_n^{(+)}\,. \tag{3}$$

On the other hand, if $1/\psi_0^{(-)}$ is normalizable, then the roles of V_- and V_+ are reversed. Consequently $E_0^{(-)}\neq 0$ whereas the ground state energy of V_+, i.e. $E_0^{(+)}=0$ [6]. If supersymmetric partner potentials satisfy the condition

$$V_+(x,a_0)=V_-(x,a_1)+R(a_1)\,,\quad a_1=f(a_0)\,, \tag{4}$$

they are called shape-invariant [5–8]. For this case, the complete set of eigenvalues and eigenfunctions can be determined by purely algebraic methods yielding

$$E_n^{(-)}=\sum_{k=1}^{n}R(a_k)\,.$$

This result is obtained using the fact that $E_0^{(-)}=0$. Until now, several types of such potentials in unbroken SUSY are known and the complete list is given in refs. [8–10].

For the case of broken supersymmetry [3,6] both V_+ and V_- have degenerate energy eigenvalues

$$E_n^{(-)}=E_n^{(+)}\,, \tag{5}$$

with ground state energies greater than zero. Relatively little attention has been paid so far to study problems involving broken SUSY. This is perhaps due to the fact that in the majority of examples cited in the literature, V_+ and V_- are related to each other by the parity operation and the degeneracy relation given by eq. (5) then follows trivially. In particular, if $W(x)$ is symmetric, it follows that $V_+(x)=V_-(-x)$. Very recently Chuan [11], Ralchenko and Semenov [12], and others [13,14] have discussed the possibility of finding connections between quantum mechanical systems with broken and unbroken supersymmetry. However, these authors did not explore the full variety of broken supersymmetric cases.

The purpose of this Letter is to examine some solvable non-trivial examples of broken SUSY. These examples are motivated by enlarging the range of the parameters of previously studied shape-invariant potentials of unbroken SUSY [8]. Our main results are presented in table 1. It is found that out of all the known types of solvable potentials, only three (three-dimensional oscillator, Pöschl–Teller I and II) admit exact analytic solutions when the parameters are extended to the domain consistent with the conditions of broken SUSY [#1]. For the remaining types, the extension of the parameters from unbroken to broken SUSY does not allow both V_- and V_+ to hold bound states.

To illustrate our procedure, we discuss the example of the Pöschl–Teller I potential in detail. For this problem the superpotential and the pair of partner potentials are [8]

$$W(x,A,B)=A\tan x-B\cot x \quad (0<x<\tfrac{1}{2}\pi), \tag{6}$$

$$V_-(x,A,B)=A(A-1)\sec^2x+B(B-1)\,\mathrm{cosec}^2x-(A+B)^2, \tag{7}$$

$$V_+(x,A,B)=A(A+1)\sec^2x+B(B+1)\,\mathrm{cosec}^2x-(A+B)^2. \tag{8}$$

It is clear from eq. (7) that V_- holds bound states for unbroken supersymmetry for two regions of the parameters. These two regions are (i) $A>1$, $B>1$ with $W(0)=-\infty$, $W(\frac{1}{2}\pi)=+\infty$, and (ii) $A<0$, $B<0$ with $W(0)=+\infty$, $W(\frac{1}{2}\pi)=-\infty$. For these cases, one obtains energy eigenvalues and eigenfunctions in an algebraic way exploiting the power of the shape-invariance condition. One thus finds [8,9]:

(i) for $A>1$, $B>1$

$$E_n^{(-)}(A,B)=(A+B+2n)^2-(A+B)^2,$$

$$\psi_n^{(-)}(y,A,B)=(1+y)^{A/2}(1-y)^{B/2}P_n^{(B-1/2,A-1/2)}(y), \tag{9}$$

(ii) for $A<0$, $B<0$

$$E_n^{(-)}(A,B)=(A+B-2n-2)^2-(A+B)^2,$$

$$\psi_n^{(-)}(y,A,B)=(1+y)^{(1-A)/2}(1-y)^{(1-B)/2}P_n^{(1/2-B,1/2-A)}(y), \tag{10}$$

where $y=\cos(2x)$ and P_n represents the Jacobi polynomial.

The results of eq. (9) were obtained earlier in refs. [8,9]. The results in eq. (10) are reported here for the first time. One may note that in this case, $E_0^{(-)}\neq 0$ but $E_0^{(+)}=0$.

The case of broken supersymmetry can be achieved by suitably changing the values of the parameters A and B. As an example, for $A<0$, $B>1$, we now show that the eigenvalues and eigenfunctions may be derived from eq. (9) using an appropriate mapping procedure.

We define $\tilde{A}=-A+1$ so that $\tilde{A}>1$ for $A<0$. In terms of $\tilde{A}$ and B, eq. (7) becomes

$$V_-(x,\tilde{A},B)=\tilde{A}(\tilde{A}-1)\sec^2x+B(B-1)\,\mathrm{cosec}^2x-(\tilde{A}+B)^2+[(\tilde{A}+B)^2-(1-\tilde{A}+B)^2]. \tag{11}$$

Clearly, except for the constant terms inside the square bracket, $V_-(x,\tilde{A},B)$ satisfies the condition (i) for unbroken SUSY since $\tilde{A}>1$, $B>1$. From eq. (9), one can write energy eigenvalues and eigenfunctions,

$$E_n^{(-)}(\tilde{A},B)=(\tilde{A}+B+2n)^2-(1-\tilde{A}+B)^2,$$

$$\psi_n^{(-)}(y,\tilde{A},B)=(1+y)^{\tilde{A}/2}(1-y)^{B/2}P_n^{(B-1/2,\tilde{A}-1/2)}(y). \tag{12}$$

[#1] The Scarf superpotential $W_{\rm Scarf}=-A\cot(\alpha x)+B\,\mathrm{cosec}(\alpha x)$ can also be included in table 1 as an example of broken SUSY. However, it can be shown that it is equivalent to the Pöschl–Teller I superpotential by appropriate re-definition of parameters.

Reverting to the original parameter A from $\tilde{A}$, one obtains the broken SUSY results for $A<0$, $B>1$,

$$E_n^{(-)}(A,B)=(B-A+2n+1)^2-(A+B)^2\,,$$

$$\psi_n^{(-)}(y,A,B)$$

$$=(1+y)^{(1-A)/2}(1-y)^{B/2}P_n^{(B-1/2,1/2-A)}(y)\,. \quad (13)$$

Starting with eq. (9) and following a similar mapping procedure, one gets the other broken supersymmetric results for $A>1$, $B<0$,

$$E_n^{(-)}(A,B)=(A-B+2n+1)^2-(A+B)^2\,,$$

$$\psi_n^{(-)}(y,A,B)$$

$$=(1+y)^{A/2}(1-y)^{(1-B)/2}P_n^{(1/2-B,A-1/2)}(y)\,. \quad (14)$$

It is interesting to see that the supersymmetric potential V_- exhibits unbroken or broken SUSY results depending on the range of the two parameters A and B. There are zones in the A–B parameter space where the potentials have a weakly attractive singular behavior and ambiguous eigenvalues [15]. These zones separate the domains of unbroken and broken SUSY. Therefore, smooth analytic continuation of parameters from unbroken zones to broken ones or vice versa is not possible. This may be seen from fig. 1. A similar observation has been made by Verbaarschot, West, and Wu [16] in their study of large order behavior of the anharmonic oscillator about the supersymmetric point.

The above exercise may be repeated for the V_+ potential. The results are displayed in fig. 2. It is evident that for the Pöschl–Teller I superpotential, V_- and V_+ can simultaneously hold bound states in the following domains of the parameters:

unbroken SUSY: $A, B>1$ and $A, B<-1$;

broken SUSY: $A<-1$, $B>1$ and $A>1$, $B<-1$.

Proceeding in the same way, we find that there are two other shape-invariant potentials (the three-dimensional harmonic oscillator and Pöschl–Teller II) which exhibit discrete eigenstates in the domain of broken supersymmetry. For these potentials, one obtains closed analytic expressions for the energy eigenvalues and eigenfunctions starting with the results obtained earlier [8,9] for unbroken SUSY. Appropriate mappings in the parameter space restore

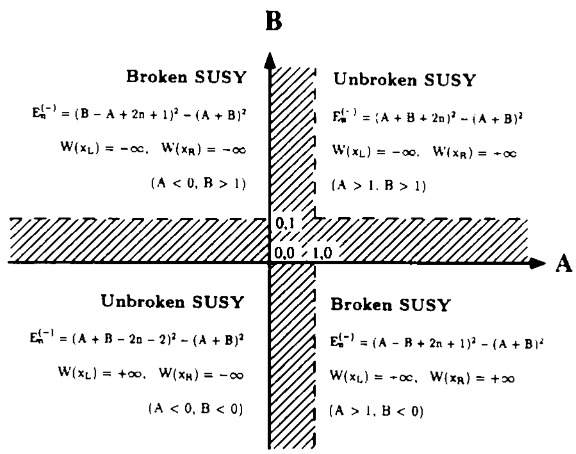

Fig. 1. The four regions of different SUSY behavior of V_- for the Pöschl–Teller I superpotential in the A–B parameter plane. For parameter values in the shaded region, the potential V_- has an attractive inverse square behavior at $x=0$ or/and $x=\frac{1}{2}\pi$, which leads to ambiguous eigenvalues characteristic of weakly attractive singular potentials. See ref. [15] for details.

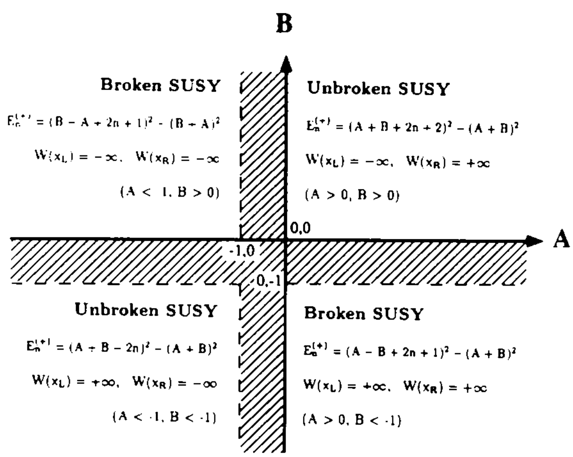

Fig. 2. The four regions of different SUSY behavior of V_+ for the Pöschl–Teller I superpotential in the A–B parameter plane. For parameter values in the shaded region, the potential V_+ has an attractive inverse square behavior at $x=0$ or/and $x=\frac{1}{2}\pi$, which leads to ambiguous eigenvalues characteristic of weakly attractive singular potentials. See ref. [15] for details.

the results in a very simple way. The final results are presented in table 1. The present work illustrates once more the utility of the shape-invariance concept for solvable quantum mechanical problems. Although we are unable to exploit directly the shape-invari-

Table 1
List of three types of superpotentials which give rise to solvable, shape-invariant partner potentials V_- and V_+. The potentials have identical energy spectra $E_n^{(-)} = E_n^{(+)}$ since they correspond to a broken supersymmetry situation.

Name of the potential	$W(r)$	$V_-(r)$	$V_+(r)$
Three-dimensional oscillator	$\frac{1}{2}\omega r-(l+1)/r$ $(0\leqslant r<\infty)$	$V_-(r,\omega,l)=\frac{1}{4}\omega^2r^2+l(l+1)/r^2-\omega(l+\frac{3}{2})$	$V_+(r,\omega,l)=V_-(r,\omega,l+1)+2\omega$
Pöschl–Teller I	$A\tan x-B\cot x$ $(0\leqslant x<\frac{1}{2}\pi)$	$V_-(x,A,B)=-(A+B)^2+A(A-1)\sec^2x$ $+B(B-1)\operatorname{cosec}^2x$	$V_+(x,A,B)=V_-(x,A+1,B+1)$ $+(A+B+2)^2-(A+B)^2$
Pöschl–Teller II	$A\tanh r-B\coth r$ $(0\leqslant r<\infty)$	$V_-(r,A,B)=(A-B)^2-A(A+1)\operatorname{sech}^2r$ $+B(B-1)\operatorname{cosech}^2r$	$V_+(r,A,B)=V_-(r,A-1,B+1)$ $+(A-B)^2-(A-B-2)^2$

ance condition (eq. (4)) for the case of broken SUSY (since $E_0^{(\pm)}\neq 0$), we have been able to make use of it indirectly by mapping the parameters in such a way that a broken SUSY problem may be viewed as an unbroken one.

This work opens up new prospective areas of investigation into broken SUSY problems. Recently, supersymmetry inspired WKB calculations have received attention in view of the fact that all shape-invariant potentials with unbroken SUSY yield exact eigenvalues in the leading order SWKB quantization condition [17,18],

$$\int_{x_1}^{x_2} \sqrt{E_n^{(-)} - W^2(x)}\, dx = n\pi\hbar\,. \qquad (15)$$

For broken supersymmetry, the corresponding condition has recently been reported [19],

$$\int_{x_1}^{x_2} \sqrt{E_n^{(\pm)} - W^2(x)}\, dx = (n+\tfrac{1}{2})\pi\hbar\,. \qquad (16)$$

Preliminary calculations show that the eigenvalues of all the potentials cited in table 1 are exact in the leading order broken supersymmetric WKB quantization (BSWKB) condition (eq. (16)). Compared to the usual WKB condition,

$$\int_{x_1}^{x_2} \sqrt{E_n^{(\pm)} - V_\pm(x)}\, dx = (n+\tfrac{1}{2})\pi\hbar\,, \qquad (17)$$

the BSWKB formula has the added advantage that it reflects the degeneracy of the spectra of $V_\pm(x)$ manifestly. Furthermore, from previous experience [20,21], we have good reason to expect that the degeneracy may exist in higher order calculations in the BSWKB framework. This is however not the case for the standard higher order WKB method [22]. In view of this, the study of non-shape-invariant broken SUSY potentials in higher order BSWKB framework will be stimulating. Such aspects, together with a number of numerical analyses for broken supersymmetric potentials in one and three dimensions will be discussed elsewhere.

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Range of parameters for broken SUSY and existence of degenerate levels	$E_n^{(-)}=E_n^{(+)}$	$\psi_n^{(-)}$
$l>0, \omega<0$	$-\omega(2n+l+\frac{3}{2})-\omega(l+\frac{3}{2})$	$r^{l+1}\exp(\frac{1}{4}\omega r^2)L_n^{l+1/2}(-\frac{1}{2}\omega r^2)$
$l<-2, \omega>0$	$\omega(2n-l+\frac{1}{2})-\omega(l+\frac{3}{2})$	$r^{-l}\exp(-\frac{1}{4}\omega r^2)L_n^{-l-1/2}(\frac{1}{2}\omega r^2)$
$A<-1, B>1$	$(B-A+2n+1)^2-(A+B)^2$	$(\cos x)^{1-A}(\sin x)^B P_n^{(B-1/2,1/2-A)}(\cos(2x))$
$A>1, B<-1$	$(A-B+2n+1)^2-(A+B)^2$	$(\cos x)^A(\sin x)^{1-B} P_n^{(1/2-B,A-1/2)}(\cos(2x))$
$A+B+1<0, B>1$	$-(A+B+2n+1)^2+(A-B)^2$	$(\sinh r)^B(\cosh r)^{A+1} P_n^{(B-1/2,A+1/2)}(\cosh(2r))$
$A+B-1>0, B<-1$	$-(A+B-2n-1)^2+(A-B)^2$	$(\sinh r)^{1-B}(\cosh r)^{-A} P_n^{(1/2-B,-A-1/2)}(\cosh(2r))$

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