IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **44** (2011) 275307 (19pp) doi:10.1088/1751-8113/44/27/275307

Method for generating additive shape-invariant potentials from an Euler equation

Jonathan Bougie, Asim Gangopadhyaya and Jeffry V Mallow

Department of Physics, Loyola University Chicago, Chicago, IL 60660, USA

E-mail: jbougie@luc.edu, agangop@luc.edu and jmallow@luc.edu

Received 6 March 2011, in final form 5 May 2011
Published 7 June 2011
Online at stacks.iop.org/JPhysA/44/275307

Abstract

In the supersymmetric quantum mechanics formalism, the shape invariance condition provides a sufficient constraint to make a quantum mechanical problem solvable, i.e. we can determine its eigenvalues and eigenfunctions algebraically. Since shape invariance relates superpotentials and their derivatives at two different values of the parameter a, it is a non-local condition in the coordinate–parameter (x, a) space. We transform the shape invariance condition for additive shape-invariant superpotentials into two local partial differential equations. One of these equations is equivalent to the one-dimensional Euler equation expressing momentum conservation for inviscid fluid flow. The second equation provides the constraint that helps us determine unique solutions. We solve these equations to generate the set of all known $\hbar$-independent shape-invariant superpotentials and show that there are no others. We then develop an algorithm for generating additive shape-invariant superpotentials including those that depend on $\hbar$ explicitly, and derive a new $\hbar$-dependent superpotential by expanding a Scarf superpotential.

PACS numbers: 45.70.Qj, 47.40.Nm, 47.54.−r

1. Introduction

1.1. Background

There are few known potentials in quantum mechanics that can be solved exactly. Many authors have attempted to systematically find and classify all such potentials [1–6]. In some of these potentials, the eigenvalues can be obtained by solving an algebraic equation [3, 4, 6]. Supersymmetric quantum mechanics (SUSYQM) provides a formalism for obtaining closed expressions for the energy eigenvalues and eigenfunctions of many one-dimensional and three-dimensional problems [7–11].

The SUSYQM formalism uses first order differential operators A^- and A^+ that are generalizations of the raising and lowering operators employed by Dirac for treating the harmonic oscillator [12]. These 'ladder' operators $A^\pm$ use a 'superpotential' W to generate 'partner Hamiltonians' $H_\pm$ with corresponding potentials $V_\pm$. If these partner potentials are 'shape invariant', then the eigenspectra for both Hamiltonians can be derived algebraically without any prior information about either Hamiltonian.

Until recently, all known shape-invariant potentials could be generated from superpotentials with no explicit dependence on $\hbar$ [10, 11]. We classify such superpotentials as 'conventional'. However, a new class of shape-invariant potentials was discovered by Quesne [13] and expanded elsewhere [14, 15]. These potentials arise from a set of 'extended' superpotentials that contain explicit $\hbar$-dependence.

In a recent publication [16], we showed that the shape-invariance condition can be transformed into two local partial differential equations. Solutions to these equations generate the set of all known conventional shape-invariant superpotentials and allow no others in this category. In addition, these equations provide an algorithm for generating $\hbar$-dependent potentials. In this paper, we elaborate on this method, proving the completeness of the set of 'conventional' superpotentials and extend it to generate a previously unknown 'extended' superpotential.

1.2. Supersymmetric quantum mechanics

For simplicity of notation, we will use units where $2m = 1$. The SUSYQM formalism makes use of first order differential ladder operators A^- and A^+:

$$A^\pm(x, a_0) = \mp\hbar\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0), \tag{1}$$

where the superpotential $W(x, a_0)$ is a real function of coordinate x and parameter a_0. $A^\pm$ are generalizations of harmonic oscillator ladder operators $\mp\hbar\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\omega x$, with $W(x, a_0)$ replacing $\frac{1}{2}\omega x$.

Operators $A^\pm$ generate two supersymmetric partner Hamiltonians: $H_- \equiv A^+A^-$ and $H_+ \equiv A^-A^+$. The Hamiltonian H_+ is called the superpartner of H_-, and corresponding potentials V_- and V_+ are given by $V_\pm = W^2(x, a_0) \pm \hbar\frac{\mathrm{d}W(x,a_0)}{\mathrm{d}x}$.

Let us denote the eigenfunctions of $H_\pm$ that correspond to eigenvalues $E_n^\pm$ by $\psi_n^{(\pm)}$. Since H_- is of the form A^+A^-, the ground state energy $E_0^{(-)} \geqslant 0$.[1] However, for $E_0^{(-)} > 0$, the supersymmetry is broken. Therefore, we only consider the case of unbroken supersymmetry, in which case $E_0^{(-)} = 0$. Thus, $A^-\psi_0^{(-)} = 0$, which is equivalent to

$$W(x, a_0) = -\hbar\left(\frac{\psi_0'}{\psi_0}\right). \tag{2}$$

For positive integer n,

$$\begin{aligned} H_+\left(A^-\psi_n^{(-)}\right) &= A^-A^+\left(A^-\psi_n^{(-)}\right) \\ &= A^-\left(A^+A^-\psi_n^{(-)}\right) \\ &= A^-H_-\left(\psi_n^{(-)}\right) \\ &= E_n^-\left(A^-\psi_n^{(-)}\right). \end{aligned} \tag{3}$$

All excited states $\psi_n^{(-)}$ of H_- have one-to-one correspondence with eigenstates of H_+ with exactly the same energy: $\psi_{n-1}^{(+)} \propto A^-\psi_n^{(-)}$, where $E_{n-1}^+ = E_n^-$ $(n = 1, 2, \ldots)$, as illustrated in figure 1. In other words, eigenstates of H_+ are iso-spectral with excited states of H_-.

[1] $E_0^{(-)} = \langle 0|A^+A^-|0\rangle = \left|A^-|0\rangle\right|^2 \geqslant 0$.

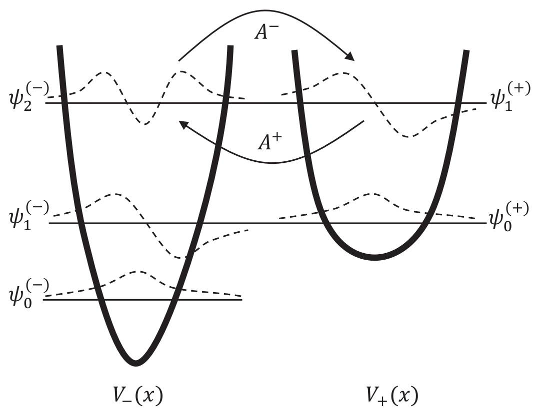

Figure 1. Schematic illustrating the isospectrality of H_+ and H_-, showing the ground state and first two excited states of H_- (corresponding to potential $V_-(x)$ sketched with a solid bold curve) in the left column, and the ground state and first excited state of H_+ (corresponding to potential $V_+(x)$, sketched as a solid bold curve) in the right column. Sample energy levels are shown as horizontal lines, and a sketch of a sample wavefunction is overlaid with a dashed line for each energy level. In general, V_+ and V_- have different shapes, as do various ψ^+ and ψ^-. However, the energy $E_0^+ = E_1^-$, $E_1^+ = E_2^-$, and $E_{n-1}^+ = E_n^-$. The operator A^- acting on an eigenstate of H_+ will yield an eigenstate of H_-, and A^+ acting on an eigenstate of H_- will yield an eigenstate of H_+.

Conversely, $\psi_n^{(-)} \propto A^+\psi_{n-1}^{(+)}$. Thus, if the eigenvalues and the eigenfunctions of H_- were known, one would automatically obtain the eigenvalues and the eigenfunctions of H_+ (and *vice versa*), which is in general a different Hamiltonian. However, unless we know one set of eigenstates *a priori*, this analysis is simply a mathematical curiosity.

1.3. Shape invariance

Despite this limitation for special cases in which these partner potentials obey the 'shape invariance' condition [1, 17, 18]:

$$V_+(x, a_0) + g(a_0) = V_-(x, a_1) + g(a_1), \tag{4}$$

where parameter a_1 is a function of a_0, i.e. $a_1 = f(a_0)$, the spectra for both Hamiltonians can be derived algebraically without any prior information for either Hamiltonian. This is due to the existence of an underlying potential algebra [19–21].

Let us consider the special case where $V_-(x, a_0)$ is a shape-invariant potential. In this case, potentials V_- and V_+ have the same x-dependence, and the corresponding Hamiltonians $H_+(x, a_0)$ and $H_-(x, a_1)$ differ by $g(a_1) - g(a_0)$. Thus, their eigenfunctions are the same, and corresponding eigenvalues differ by $g(a_1) - g(a_0)$. In particular, they have a common ground state wavefunction, given by $\psi_0^{(+)}(x, a_0) = \psi_0^{(-)}(x, a_1) \sim \exp\left(-\int_{x_0}^{x} W(x, a_1)\,\mathrm{d}x\right)$, and the ground state energy of $H_+(x, a_0)$ is $g(a_1) - g(a_0)$, because the ground state energy of $H_-(x, a_1)$ is zero. Note that the parameter shift $a_0 \to a_1$ has an effect similar to that of a ladder operator: $\psi_1^{(-)}(x, a_0) \sim A^+(x, a_0)\psi_0^{(-)}(x, a_1)$. Also note that the ladder operators A^- and A^+, like $H^{\pm}$, are dependent on parameters a_n.

The first excited state of $H_-(x, a_0)$ is given to within normalization by $A^+(x, a_0)\psi_0^{(-)}(x, a_1)$ and the corresponding eigenvalue is $g(a_1) - g(a_0)$. By iterating this procedure, the $(n + 1)$st excited state is given by

$$\psi_{n+1}^{(-)}(x, a_0) \sim A^+(a_0)A^+(a_1)\cdots A^+(a_n)\psi_0^{(-)}(x, a_n)\ , \tag{5}$$

and corresponding eigenvalues are given by

$$E_0^- = 0 \quad \text{and} \quad E_n^- = g(a_n) - g(a_0) \quad \text{for} \quad n > 0. \tag{6}$$

(To avoid notational complexity, we have suppressed the x-dependence of operators $A(x, a_0)$ and $A^+(x, a_0)$.) Thus, for a shape-invariant potential, one can obtain the entire spectrum of H_- itself by the algebraic methods of SUSYQM (and of course the same is true for H_+).

In this paper, we develop a method for finding shape-invariant superpotentials from a system of partial differential equations, and use this method to discover a new shape-invariant superpotential. In section 2, we will show that for $\hbar$-independent ('conventional') superpotentials, the shape invariance condition can be converted into an infinite sequence of partial differential equations. In section 3, we solve these partial differential equations and systematically generate the complete set of conventional shape-invariant potentials. In section 4, we show a method for using these equations to find $\hbar$-dependent ('extended') superpotentials. In section 5, we use this method to find a new shape-invariant superpotential, and we present our conclusions in section 6.

2. Expressing the shape invariance condition with partial differential equations for conventional superpotentials

2.1. Shape invariance expressed by a difference–differential equation

As we have just seen, any partner potentials obeying (4) can be solved algebraically. Thus, discovering that a potential is shape invariant yields much useful information. Writing (4) in terms of the superpotential, we obtain

$$W^2(x, a_0) + \hbar\,\frac{\partial W(x, a_0)}{\partial x} + g(a_0) = W^2(x, a_1) - \hbar\,\frac{\partial W(x, a_1)}{\partial x} + g(a_1). \tag{7}$$

For shape-invariant systems, the energy eigenvalues of $H_-(x, a_0)$ are given by $E_n^{(-)}(a_0) = g(a_n) - g(a_0)$, where $a_n \equiv f^n(a_0)$ indicates f applied n-times to a_0 [10, 11]. To avoid level crossing, $g(a)$ must be a monotonically increasing function of a, i.e. $\frac{\partial g}{\partial a} > 0$. In the case of 'additive' or 'translational' shape invariance, the parameters differ by an additive constant, i.e. $a_{i+1} = a_i + \hbar$. There are other forms of shape invariance such as multiplicative [22] and cyclic [23]. Spiridonov in particular [24] studied scaling of the coordinate x as well as the superpotential W. His work was shown to be equivalent to multiplicative shape invariance [25]. Most of the known exactly solvable superpotentials exhibit additive shape invariance [5, 26]. We thus restrict our analysis to additive shape invariance.

Several groups have found the conventional shape-invariant superpotentials by imposing various ansätze [5, 21, 27, 28]. For a multiple parameter extension, see [29]. In this paper, we derive these potentials *ab initio* as the solutions to a set of partial differential equations that must be satisfied for all additive shape-invariant potentials.

Note that (7) is a difference–differential equation; that is, it relates the square of the superpotential W and its spatial derivative computed at two different parameter values: $(x, a_0 \equiv a)$ and $(x, a_1 \equiv a + \hbar)$. This equation has also been studied for dynamical systems, where it is known as the infinite-dimensional dressing chain [30]. Furthermore, in (7) we have not needed to specify the value of $\hbar$; therefore, this equation must hold for any value.

Using this feature, we will transform (7) into a set of nonlinear partial differential equations that are local in nature, i.e. all terms can be computed at the same point (x, a). This has the obvious advantage of mathematical familiarity (at least to physicists). In addition, it provides a systematic method for finding additive shape-invariant potentials.

2.2. *Shape invariance expressed by differential equations*

Important correspondences have been shown to exist between quantum mechanics and fluid mechanics [31]. SUSYQM is well known to have a deep connection with the KdV equation [32–36], a nonlinear equation that describes waves in shallow water. In this section, we show that every additive shape-invariant superpotential that does not depend explicitly on $\hbar$ corresponds to a solution of the Euler equation expressing momentum conservation for inviscid fluid flow in one spatial dimension. We use this correspondence to develop a systematic method which (1) yields all such known $\hbar$-independent solvable potentials for SUSYQM and (2) shows that no others exist. We will consider $\hbar$-dependent potentials in section 4.

Due to additive shape invariance, the dependence of W on a and $\hbar$ is through the linear combination $a + \hbar$; therefore, the derivatives of W with respect to a and $\hbar$ are related by $\frac{\partial W(x,a+\hbar)}{\partial\hbar} = \frac{\partial W(x,a+\hbar)}{\partial a}$. Since (7) must hold for an arbitrary value of $\hbar$, if we assume that W does not depend explicitly on $\hbar$, we can expand in powers of $\hbar$ and the coefficient of each power must separately vanish. Expanding the right-hand side in powers of $\hbar$ yields

$$\mathcal{O}(\hbar) \Rightarrow W\,\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x} + \frac{1}{2}\,\frac{\mathrm{d}g(a)}{\mathrm{d}a} = 0, \tag{8}$$

$$\mathcal{O}(\hbar^2) \Rightarrow \frac{\partial}{\partial a}\left(W\,\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x} + \frac{1}{2}\,\frac{\mathrm{d}g(a)}{\mathrm{d}a}\right) = 0, \tag{9}$$

$$\mathcal{O}(\hbar^n) \Rightarrow \frac{\partial^n}{\partial a^{n-1}\partial x} W(x,a) = 0, \qquad n \geqslant 3. \tag{10}$$

Thus, all conventional additive shape-invariant superpotentials are solutions of the above set of nonlinear partial differential equations. Although this represents an infinite set, note that if equations at $\mathcal{O}(\hbar)$ and $\mathcal{O}(\hbar^3)$ are satisfied, all others automatically follow. In section 3, we will find the complete set of solutions of these equations.

2.3. *Connection to fluid mechanics*

Replacing W by $-u$, x by t, and a by x in (8), the equation then becomes

$$\left(u(x,t)\,\frac{\partial}{\partial x}\right)u(x,t) + \frac{\partial u(x,t)}{\partial t} = -\frac{1}{2}\,\frac{\mathrm{d}g(x)}{\mathrm{d}x}. \tag{11}$$

This equation is equivalent to the equation for inviscid fluid flow in the absence of an external force on the bulk of the fluid:

$$\frac{\partial \mathbf{u}\,(\mathbf{x},t)}{\partial t} + \mathbf{u}\,(\mathbf{x},t)\cdot\nabla\mathbf{u}\,(\mathbf{x},t) = -\frac{\nabla p\,(\mathbf{x},t)}{\rho\,(\mathbf{x},t)} \tag{12}$$

in one spatial dimension with the correspondence $\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}x}$, where $\mathbf{u}$ is the fluid velocity at location $\mathbf{x}$ and time t, p is the pressure, and ρ is the local fluid density. Equation (12) is one of the fundamental laws of fluid dynamics and was first obtained by Euler in 1755 [37]. Thus, all conventional shape-invariant superpotentials form a set of solutions to the one-dimensional Euler equation.

It should be noted that (12) is not solvable unless additional constraints are applied. In fluid dynamics, this equation is generally supplemented by the continuity equation expressing conservation of mass, along with an equation of state and/or the energy equation and boundary conditions. These additional constraints do not apply in SUSYQM. Instead, (10) supplies the additional constraint which must be fulfilled to satisfy shape invariance. Thus, the set of solutions of (8) that also satisfy the constraint of (10) will define the complete set of conventional shape-invariant superpotentials.

3. Generating the complete set of conventional superpotentials

3.1. Solutions of special cases

In this section, we show that the set of all possible conventional superpotentials are determined by six special cases. To find this set of solutions, we note that (10) is satisfied for all $n \geqslant 3$ as long as

$$\frac{\partial^3}{\partial a^2 \partial x} W(x,a) = 0. \tag{13}$$

The general solution to (13) is

$$W(x,a) = a \cdot X_1(x) + X_2(x) + u(a). \tag{14}$$

Substituting this into (8) yields

$$\underbrace{(a \cdot X_1 + X_2 + u)}_{W} \underbrace{\left(X_1 + \frac{\mathrm{d}u}{\mathrm{d}a}\right)}_{\frac{\partial W}{\partial a}} - \underbrace{\left(a \cdot \frac{\mathrm{d}X_1}{\mathrm{d}x} + \frac{\mathrm{d}X_2}{\mathrm{d}x}\right)}_{\frac{\mathrm{d}W}{\mathrm{d}x}} + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}a} = 0. \tag{15}$$

To systematically find all possible solutions, we define $u\,\frac{\mathrm{d}u}{\mathrm{d}a} + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}a} = -H(a)$, and then collect and label terms based on their dependence on X_1 and X_2 and their derivatives:

$$\underbrace{X_1\, X_2}_{\text{Term\#1}} + \underbrace{\left(-\frac{\mathrm{d}X_2}{\mathrm{d}x}\right)}_{\text{Term\#2}} + \underbrace{a\, X_1^2}_{\text{Term\#3}} + \underbrace{\left(-a\frac{\mathrm{d}X_1}{\mathrm{d}x}\right)}_{\text{Term\#4}} + \underbrace{\frac{\mathrm{d}u}{\mathrm{d}a}\, X_2}_{\text{Term\#5}} + \underbrace{\left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right) X_1}_{\text{Term\#6}} = H(a). \tag{16}$$

We begin by considering special cases of (16) where one or more of the terms $X_1(x)$, $X_2(x)$, or u can be considered to be zero. We will then show that all solutions to this equation can be reduced to one of these special cases. In the nomenclature that follows, lowercase Greek letters denote a- and x-independent constants.

3.1.1. Case 1: X_2 and u are not constants, X_1 is constant. In this case, let $X_1 = \mu$. Then, our general form for W becomes $W = \mu a + u(a) + X_2(x)$. If we define $\tilde{u} \equiv u(a) + \mu a$, we get $W = \tilde{u} + X_2$. So this case is equivalent to $X_1 = 0$. Then terms 1, 3, 4, and 6 each become zero, and (16) becomes $-\frac{\mathrm{d}X_2}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}a} X_2 = H(a)$. Since X_2 must be independent of a but cannot be constant, $\frac{\mathrm{d}X_2}{\mathrm{d}x} \neq 0$. Thus, this is possible only if $\frac{\mathrm{d}u}{\mathrm{d}a}$ and $H(a)$ are independent of a. This yields $u = \alpha a + \beta$, $H(a) = \theta$. Therefore, $-\frac{\mathrm{d}X_2}{\mathrm{d}x} + \alpha X_2 = \theta$, where $\alpha \neq 0$ since u is not constant. The solution to this differential equation is $X_2(x) = \frac{\theta}{\alpha} + \eta\, \mathrm{e}^{\alpha x}$. Therefore, $W = \alpha a + \beta + \frac{\theta}{\alpha} + \eta\, \mathrm{e}^{\alpha x}$. Defining $\alpha = -1$, this yields $W = A - B\mathrm{e}^{-x}$, where $A \equiv \beta - a - \theta$. This is the Morse superpotential.

3.1.2. Case 2: X_1 and u are not constants, X_2 is constant. Following a similar procedure, this case is equivalent to $X_2 = 0$. So $aX_1^2 - a\frac{\mathrm{d}X_1}{\mathrm{d}x} + \left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right)X_1 = H(a)$. Since X_1 cannot depend on a but must contain x-dependence, the only way for $H(a)$ to be independent of x is if $u + a\frac{\mathrm{d}u}{\mathrm{d}a} = \alpha a$, where α could be any constant. Thus, $u = \frac{\alpha a}{2} + \frac{\beta}{a}$, where α and β could be any two constants, although they cannot both be zero.

Thus, $aX_2^2 - a\frac{\mathrm{d}X_1}{\mathrm{d}x} + a\alpha X_1 = H(a)$. This is only possible if $H(a) = \gamma a$ for some constant γ. So $X_1^2 - \frac{\mathrm{d}X_1}{\mathrm{d}x} + \alpha X_1 = \gamma$. This differential equation gives different solutions depending on the values of α and γ.

If $\alpha \neq 0$, then $\gamma \neq 0$ yields the Rosen–Morse I superpotential, and $\gamma = 0$ yields the Eckart superpotential.

If $\alpha = 0$, then $\gamma = 0$ yields Coulomb, $\gamma > 0$ yields Rosen–Morse II, and $\gamma < 0$ yields Rosen–Morse I.

Thus, the Rosen–Morse I, Rosen–Morse II, Eckart, and Coulomb superpotentials are all solutions to case 2 for different values of α and γ.

3.1.3. Case 3: X_1 and X_2 are not constants, $u = \mu a + \nu$ (this includes the case where u is constant). In this case we can define $\tilde{X}_1 \equiv X_1 + \mu$ and $\tilde{X}_2 \equiv X_2 + \nu$, which is equivalent to $u = 0$. For $u = 0$, $X_1X_2 - \frac{\mathrm{d}X_2}{\mathrm{d}x} + a\left(X_1^2 - \frac{\mathrm{d}X_1}{\mathrm{d}x}\right) = H(a)$.

Since X_1 and X_2 are both independent of a, the coefficients of each power of a must cancel separately. So we are left with two coupled differential equations: $X_1^2 - \frac{\mathrm{d}X_1}{\mathrm{d}x} = \alpha$, $X_1X_2 - \frac{\mathrm{d}X_2}{\mathrm{d}x} = \beta$. Again, the solution varies depending on the values of α and β, which could be any constants.

If $\alpha = 0$, the solution is the 3D harmonic oscillator superpotential. If $\alpha < 0$, the solution is the Scarf I superpotential, and if $\alpha > 0$, the equation is solved by either the Scarf II or generalized Pöschl–Teller superpotential.

Therefore, the Scarf I, Scarf II, 3D oscillator, and generalized Pöschl–Teller superpotentials are all solutions of case 3.

3.1.4. Case 4: X_2 is not constant, X_1 and u are constant. If $X_1 \neq 0$, this is equivalent to $X_1 = 0$ and $u = \mu a + \nu$. In such a case, this is equivalent to case 1, and the solution is the Morse superpotential. However, if $X_1 = 0$, this case is equivalent to $X_1 = u = 0$, in which case $-\frac{\mathrm{d}X_2}{\mathrm{d}x} = H(a)$. Since X_2 is independent of a, $H(a)$ must be a constant. This generates the one-dimensional harmonic oscillator.

3.1.5. Case 5: X_1 is not constant, X_2 and u are constant. In this case, $\alpha X_1 + aX_1^2 - a\frac{\mathrm{d}X_1}{\mathrm{d}x} = H(a)$ for some constant α. Since X_1 is independent of a but must depend on x, $\alpha = 0$ and $H(a) = \beta a$ for some constant β. Thus, $X_1^2 - \frac{\mathrm{d}X_1}{\mathrm{d}x} = \beta$. This yields special cases of Scarf I and Scarf II, and the centrifugal term of the Coulomb and 3D oscillator, depending on whether β is positive, negative, or zero.

3.1.6. Case 6: X_1 is constant, X_2 is constant. In this case, the superpotential has no x-dependence, regardless of the value of u. This is a trivial solution corresponding to a flat potential, and we disregard it.

Thus far, we have considered all of the special cases that are equivalent to $u = 0$, $X_1 = 0$, or $X_2 = 0$. These special cases generate all known conventional additive shape-invariant superpotentials [10, 11, 26]. We have listed them all in table 1.

Table 1. The complete family of additive shape-invariant superpotentials. Each superpotential can be obtained from a special case where at least one of the terms X_1, X_2, or u is zero in (14), as indicated in the third column.

Name	Superpotential	Special cases
Harmonic oscillator	$\frac{1}{2}\omega x$	$X_1 = u = 0$
Coulomb	$\frac{e^2}{2(\ell+1)} - \frac{\ell+1}{r}$	$X_2 = 0$
3D oscillator	$\frac{1}{2}\omega r - \frac{\ell+1}{r}$	$u = 0$
Morse	$A - Be^{-x}$	$X_1 = 0$
Rosen–Morse I	$-A\cot x - \frac{B}{A}$	$X_2 = 0$
Rosen–Morse II	$A\tanh x + \frac{B}{A}$	$X_2 = 0$
Eckart	$-A\coth x + \frac{B}{A}$	$X_2 = 0$
Scarf I	$A\tan x - B\sec x$	$u = 0$
Scarf II	$A\tanh x + B\,\mathrm{sech}\, x$	$u = 0$
Generalized Pöschl–Teller	$A\coth x - B\,\mathrm{cosech}\, x$	$u = 0$

3.2. Proof that the list of conventional superpotentials is complete

Now that we have these special cases, we can systematically obtain all possible solutions. $H(a)$ is independent of x. Therefore, when any solution is substituted into (16), it will yield an x-independent sum of terms 1–6. There are many ways in which these terms could add up to a term independent of x. As a first step, we begin with the simplest possibility, in which each term is individually independent of x.

Under this assumption, term 3 states that X_1 must be a constant, independent of x. In addition, term 1 dictates X_1X_2 must be constant as well. These two statements can only be true if X_2 and X_1 are constant separately; this reduces to the trivial solution of case 6.

Therefore, assuming that each term is separately independent of x yields only the trivial solution. However, there is also the possibility that some of the terms depend on x, but when added to other terms, the x-dependence cancels to yield a sum that is independent of x. If a group of n-terms taken together produce an x-independent sum, and if no smaller subset of these terms add up to a sum independent of x, we call this group of n-terms 'irreducibly independent of x'.

As an example, we check whether there are any solutions for the six-term irreducible set in which the sum of all six terms in (16) is independent of x, but in which the sum of any subset of terms would depend on x.

To check this possibility, we note that the first two terms are independent of a, while terms 3 and 4 are linear in a. We do not know *a priori* the functional form of u. Since term 5 contains $\frac{du}{da}$, it could include terms independent of a, terms linear in a, and/or other forms of a-dependence. Similarly, we do not know the functional form of $u + a\frac{du}{da}$. Therefore, we define functions $f_1(a)$ and $f_2(a)$ such that $u + a\frac{du}{da} = \alpha_1 + \beta_1 a + f_1(a)$ and $\frac{du}{da} = \alpha_2 + \beta_2 a + f_2(a)$, where we do not know the functional form of $f_1(a)$ and $f_2(a)$, except that they contain no constant terms or terms linear in a.

With this definition, term 6 becomes $\alpha_1 X_1 + a\beta_1 X_1 + f_1(a)X_1$ and term 5 becomes $\alpha_2 X_2 + a\beta_2 X_2 + f_2(a)X_2$. Since $f_1(a)$ and $f_2(a)$ contain no constant terms or terms proportional to a, the x dependence of the term $f_1(a)X_1$ cannot be canceled by terms 1–4, and neither can the x-dependence of $f_2(a)X_2$. Therefore, we conclude that $f_1(a)X_1 + f_2(a)X_2$ is independent of x.

Table 2. Assuming that a single term is independent of x results in restrictions on the possible solutions. The first column lists each possible single term set. The middle column of each row shows what consequences are necessary for that set to be independent of x. Throughout this section, lower-case Greek letters indicate constants that are independent of both a and x. Finally, if this requirement is compatible with solutions not included as one of the special cases, restrictions on such a solution are shown in the third column of that row. The third column states 'No' if the requirement is equivalent to a special case and thus no new solutions are allowed.

Single term set	Requirements for x-independence	Compatible with a new solution?
{1}	$X_1 = \alpha/X_2$	If $\alpha \neq 0$
{2}	$X_2 = \alpha x + \beta$	If $\alpha \neq 0$
{3}	$X_1 = \alpha$	No
{4}	$X_1 = \alpha x + \beta$	If $\alpha \neq 0$
{5}	$X_2 = \alpha$	No
{6}	$u = \alpha/a$	If $\alpha \neq 0$

This leaves only two possibilities. First we consider the possibility that $f_1(a) = f_2(a) = 0$. In this case, $\frac{\mathrm{d}u}{\mathrm{d}a} = \alpha_2 + \beta_2 a$, so $u = \alpha_2 a + \frac{\beta_2 a^2}{2} + \gamma$. Substituting this solution into (16), term 6 then becomes $\left(\frac{3\beta_2}{2}a^2 + 2\alpha_2 + \gamma\right)X_1$. Since X_1 is independent of a and none of the other terms are proportional to a^2, $\frac{3\beta_2}{2}X_1$ must be independent of x. So either X_1 is a constant, or $\beta_2 = 0$, in which case u is linear in a; either possibility represents a special case.

On the other hand, if $f_1(a) \neq 0$, then either X_1 is a constant (which is a special case), or the x-dependence of $f_1(a)X_1$ must be canceled by the x-dependence of $f_2(a)X_2$. Since X_1 and X_2 cannot depend on a, this is only possible if $X_2 + \mu X_1 = \nu$ for some constants μ and ν. However, if this is the case, then $W = aX_1 + X_2 + u = aX_1 - \mu X_1 + \nu + u$. Since the zero of a can be shifted by defining $\tilde{a} = a - \mu$, and we can define $\tilde{u} = u + \nu$, this is equivalent to the case $W = \tilde{a}X_1 + \tilde{u}$. So this particular case can be reduced to the special case of $X_2 = 0$ and does not yield any new solutions. Thus, no new solutions can be found by assuming that the full six-term set is irreducibly independent of x; any solution found under this assumption can be reduced to one of the special cases.

Since each term cannot be separately independent of x and the full six-term set cannot be irreducibly independent of x, the only possibility that could produce solutions not covered under one of the special cases would be if two or more irreducibly independent sets combine to produce a solution. If, for example, term 5 depends on x and term 6 depends on x, but the sum of these two terms is x-independent, then we consider the set of terms {5, 6} to be a two-term set that is irreducibly independent of x. To continue this example, if the four sets {1}, {2, 3}, {4}, and {5, 6} were each irreducibly independent of x, then the entire left side of (16) would be independent of x, and this could possibly produce a solution not covered under the special cases. We now proceed to examine all possible combinations and show that no new solutions are, in fact, produced.

To examine all possibilities, we begin by examining all possible one-, two-, and three-term sets, and see what restrictions are imposed on solutions in each case. For example, if term 1 is individually independent of x, then $X_1 X_2 = \alpha$ for some constant α, which implies that $X_1 = \alpha/X_2$. Table 2 shows the consequences required for each possible one-term set to be independent of x.

We now proceed to examine two-term sets. Let us take the example mentioned above in which {5, 6} is a two-term set that is irreducibly independent of x. We consider this example further to see if it leads to any new solutions. In this example, $\frac{\mathrm{d}u}{\mathrm{d}a}X_2 + \left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right)X_1$ must be independent of x. Since X_1 and X_2 must each depend on x and $\frac{du}{da}$ must depend on a (or this would reduce to a special case), the only way this is possible is if the x-dependence of $\frac{du}{da}X_2$ is canceled by the x-dependence of $\left(u + a\frac{du}{da}\right) X_1$. For this to be true, the a-dependence of $\frac{du}{da}$ must differ from the a-dependence of $u + a\frac{du}{da}$ by only a multiplicative constant. Thus, we conclude that $X_2 + \alpha X_1 = \beta$ for some constants α and β. As we did when considering the full six-term set, we can absorb X_2 into X_1 by shifting the zero of a. So this particular case can be reduced to the special case of $X_2 = 0$ and does not yield any new solutions.

However, not all two-term sets can be reduced in this way. As an example, we consider the set {1, 2}. For this set to be irreducibly independent of x, $X_1 X_2$ and $\frac{dX_2}{dx}$ must each depend on x. However, $X_1 X_2 - \frac{dX_2}{dx}$ must be independent of x. Since X_1 and X_2 do not depend on a, the only way for this to be possible is if $X_1 X_2 - \frac{dX_2}{dx} = \alpha$ for some constant α. By itself, this requirement could allow many solutions. Therefore, we will have to check whether this set can couple with the remaining terms 3, 4, 5, and 6 in such a way as to produce new solutions that are not compatible with the special cases.

In table 3, we display the consequences required for each possible two-term set to be irreducibly independent of x. From this table, it is clear that there are several two-term sets that could be irreducibly independent of x. However, in order to satisfy (16), these sets must be compatible with solutions that allow the remaining four terms to be independent of x. We now check whether there are any combinations of one- and two-term sets that can satisfy (16). Once we have completed this, we will consider cases involving three-, four-, and five-term irreducible sets.

Combining the results from tables 2 and 3 allows us to immediately eliminate many possibilities. For instance, in the example listed above, the set {1, 2} is irreducibly independent if $X_1 X_2 - \frac{dX_2}{dx} = \alpha$ for some constant α. However, in order to satisfy (16), the remaining terms must combine in such a way that the combination of terms 3, 4, 5, and 6 are independent of x as well. We first note from table 2 that term 3 cannot be independent of x by itself, and neither can term 5. From table 3, we note that {3, 6} cannot be irreducibly independent of x, and the set {5, 6} is equivalent to a special case.

Therefore, there are only two possible combinations of one- and two-term sets involving the irreducible set {1, 2} that could satisfy (16) and lead to a new solution. The first possibility is that set {1, 2} is irreducibly independent of x, set {3, 5} is irreducibly independent of x, and set {4, 6} is irreducibly independent of x. However, from table 3, {3, 5} requires $u = \frac{\alpha a^2}{2} + \gamma$ with $\alpha \neq 0$. On the other hand, {4, 6} requires $u = \frac{\alpha}{a} + \gamma$, so these two terms are incompatible. The only other possibility is that set {1, 2} is irreducibly independent of x, set {3, 5} is irreducibly independent of x, and sets {4} and {6} are each separately independent of x. However, once again the requirement $u = \alpha/a$ for term 6 to be x-independent is incompatible with the requirement $u = \frac{\alpha a^2}{2} + \gamma$ with $\alpha \neq 0$ for {3, 5}. Therefore, there can be no possible new solutions resulting from combinations of one- and two-term sets involving the irreducible set {1, 2}.

Comparing tables 2 and 3, every possible combination of one- and two-term sets leads to one of the following three results: (a) one of the terms is directly equivalent to a special case (such as any combination involving the single-term set {3}); (b) one of the terms leads to a contradiction in which a term assumed to be irreducible must be reducible (such as any combination involving the two-term set {1, 3}); or (c) two elements in the combination require different functional forms of u, leading to the impossibility of a common solution (such as any combination involving the sets {3, 5} and {4, 6}). Therefore, we conclude that no new solutions can be found from combinations of one- and two-term sets.

Table 3. Assuming that a two-term set is irreducibly independent of x results in restrictions on the possible solutions. The first column lists each possible two-term set. The middle column of each row shows what consequences are necessary for that set to be irreducibly independent of x. The middle column states 'Contradiction: must reduce' if the assumption that the two-term set is irreducibly independent of x leads to the contradictory conclusion that the set must be reducible. Finally, if this requirement is compatible with solutions not included as one of the special cases, restrictions on such a solution are shown in the third column of that row. The third column states 'No' if no new solutions are allowed, either because the assumption of irreducibility leads to a contradiction or because the requirement on solutions is equivalent to a special case.

Two-term set	Requirements for irreducible x-independence	Compatible with a new solution?
$\{1, 2\}$	$X_1X_2 - \frac{dX_2}{dx} = \alpha$	For any α
$\{1, 3\}$	Contradiction: must reduce	No
$\{1, 4\}$	Contradiction: must reduce	No
$\{1, 5\}$	$u = \mu a + \nu$	No
$\{1, 6\}$	$u = \frac{\alpha}{a} + \gamma$; $X_1 = \frac{\beta}{X_2+\alpha}$	If $\alpha \neq 0$; $\beta \neq 0$
$\{2, 3\}$	Contradiction: must reduce	No
$\{2, 4\}$	Contradiction: must reduce	No
$\{2, 5\}$	$u = \mu a + \nu$	No
$\{2, 6\}$	$u = \frac{\alpha}{a} + \gamma$; $-\frac{dX_2}{dx} + \alpha X_1 = \beta$	If $\alpha \neq 0$
$\{3, 4\}$	$X_1^2 - \frac{dX_1}{dx} = \alpha$	For any α
$\{3, 5\}$	$u = \frac{\alpha a^2}{2} + \gamma$; $X_1^2 + \alpha X_2 = \beta$	If $\alpha \neq 0$
$\{3, 6\}$	Contradiction: must reduce	No
$\{4, 5\}$	$u = \frac{\alpha a^2}{2} + \gamma$; $\frac{dX_1}{dx} = \alpha X_2 + \beta$	If $\alpha \neq 0$
$\{4, 6\}$	$u = \frac{\alpha}{a} + \gamma$; $X_1 = \beta e^{\alpha x} + \mu$	If $\alpha \neq 0$; $\beta \neq 0$
$\{5, 6\}$	$X_2 = \alpha X_1 + \beta$	No

However, there is still the possibility that there are solutions provided by combinations involving three-, four-, or five-term irreducible sets. We now test these possibilities, beginning with three-term sets. In table 4, we display the consequences required for each possible three-term set to be irreducibly independent of x.

Following the same procedure as for two-term sets, we see that once again all combinations of one-, two-, and three-term sets produce one of the following results: (a) one of the terms is directly equivalent to a special case; (b) one of the terms leads to a contradiction in which a term assumed to be irreducible must be reducible; or (c) two elements in the combination require different functional forms of u, leading to the impossibility of a common solution.

We now turn to the possibility of solutions involving four-term irreducible sets. Rather than calculating the restrictions on all possible four-term sets, we use the fact that each four-term set must combine with either a two-term set or a pair of single-term sets. We can use this fact to eliminate many possibilities. For instance, in order for the set $\{2, 4, 5, 6\}$ to yield new solutions, it must combine either with the two-term set $\{1, 3\}$ or with the pair of single-term

Table 4. Assuming that a three-term set is irreducibly independent of x results in restrictions on the possible solutions. The first column lists each possible three-term set. The middle column shows the consequences necessary for that set to be irreducibly independent of x. If this requirement is compatible with solutions not included as one of the special cases, restrictions on such a solution are shown in the third column.

Three-term set	Requirements for irreducible x-independence	Compatible with a new solution?
{1, 2, 3}	Contradiction: must reduce	No
{1, 2, 4}	Contradiction: must reduce	No
{1, 2, 5}	$u = \mu a + \nu$	No
{1, 2, 6}	$u = \frac{\alpha}{a} + \gamma$; $X_1 X_2 - \frac{dX_2}{dx} + \alpha X_1 = \mu$	If $\alpha \neq 0$
{1, 3, 4}	Contradiction: must reduce	No
{1, 3, 5}	Contradiction: must reduce	No
{1, 3, 6}	Contradiction: must reduce	No
{1, 4, 5}	$u = \frac{\alpha a^2}{2} + \beta a + \gamma$; $X_2 = \frac{\nu}{X_1+\beta}$; $-\frac{dX_1}{dx} + \frac{\alpha\nu}{X_1+\beta} = \mu$	If $\alpha \neq 0; \beta \neq 0$
{1, 4, 6}	$u = \frac{\alpha a}{2} + \beta + \frac{\gamma}{a}$ $X_1 = \frac{\nu}{X_2+\beta}$ $X_1 = \mu e^{\alpha x} + \eta$	$\alpha \neq 0; \beta \neq 0$
{1, 5, 6}	$X_2 = \alpha X_1 + \beta$	No
{2, 3, 4}	Contradiction: must reduce	No
{2, 3, 5}	$u = \frac{\alpha a^2}{2} + \beta a + \gamma$; $X_2 = \mu e^{\beta x} + \nu$; $X_1^2 + \alpha X_2 = \mu$	If $\alpha \neq 0; \beta \neq 0$; $\mu \neq 0$
{2, 3, 6}	Contradiction: must reduce	No
{2, 4, 5}	$u = \frac{\alpha a^2}{2} + \beta a + \gamma$; $X_2 = \mu e^{\beta x} + \nu$; $-\frac{dX_1}{dx} + \alpha X_2 = \eta$	If $\alpha \neq 0; \beta \neq 0$ $\mu \neq 0$
{2, 4, 6}	$u = \frac{\alpha a}{2} + \beta + \frac{\gamma}{a}$; $X_2 = \mu e^{\beta x} + \nu$; $-\frac{dX_2}{dx} + \beta X_1 = \eta$	If $\alpha \neq 0; \beta \neq 0$ $\mu \neq 0$
{2, 5, 6}	$X_2 = \alpha X_1 + \beta$	No
{3, 4, 5}	Contradiction: must reduce	No
{3, 4, 6}	$u = \frac{\alpha a-\gamma}{2} + \frac{\gamma}{a}$ $X_1^2 - \frac{dX_1}{dx} + \alpha X_1 = \mu$	If $\alpha \neq 0$
{3, 5, 6}	Contradiction: must reduce	No
{4, 5, 6}	$X_2 = \alpha X_1 + \beta$	No

sets {1} and {3}. Since neither possibility can yield new solutions, we can eliminate the four-term set {2, 4, 5, 6} from consideration. As a different example, term 5 must depend on x (cf table 2), so the set {1, 2, 3, 6} cannot combine with the single terms {4} and {5}. However, it may still be possible for {1, 2, 3, 6} to be irreducibly independent of x if the two-term set {4, 5} is irreducibly independent of x. In table 5, we display a list of the four-term sets. For each four-term set, we list the two-term set or pair of single-term sets with which it must combine.

Of the remaining four-term sets, many more can be considered by assuming the requirements from the complementary one- or two-term sets and substituting these solutions

Table 5. For a four-term set to be irreducibly independent of x, either the two remaining terms must each be separately independent of x or the combination of the two remaining terms must produce an irreducibly independent two-term set. The first column of this table lists all possible four-term sets. The second column shows the pair of one-term sets that could combine with this four-term set; if the pair of one-term sets cannot lead to a new potential according to table 2, this column reads 'none'. The final column shows the two-term set that could combine with this four-term set. If this two-term set cannot lead to a new solution according to table 3, this column reads 'none'.

Four-term set	Eligible combination of one-term partners	Eligible two-term partner
{1, 2, 3, 4}	None	None
{1, 2, 3, 5}	{4}, {6}	{4, 6}
{1, 2, 3, 6}	None	{4, 5}
{1, 2, 4, 5}	None	None
{1, 2, 4, 6}	None	{3, 5}
{1, 2, 5, 6}	None	{3, 4}
{1, 3, 4, 5}	{2}, {6}	{2, 6}
{1, 3, 4, 6}	None	None
{1, 3, 5, 6}	{2}, {4}	None
{1, 4, 5, 6}	None	None
{2, 3, 4, 5}	{1}, {6}	{1, 6}
{2, 3, 4, 6}	None	None
{2, 3, 5, 6}	{1}, {4}	None
{2, 4, 5, 6}	None	None
{3, 4, 5, 6}	{1}, {2}	{1, 2}

into the remaining equation. For instance, according to table 5, the set {1, 2, 3, 6} can be irreducibly independent of x only if {4, 5} is irreducibly independent of x as well. However, from table 3, this would require $u = \frac{\alpha a^2}{2} + \gamma$. Plugging this solution into (16) requires that $X_1 X_2 - \frac{\mathrm{d}X_2}{\mathrm{d}x} + \gamma X_1 + a X_1^2 + \frac{3\alpha a^2}{2} X_1$ be independent of x. However, as aX_1^2 is the only term linear in a, the only solution is that X_1 be constant, which leads to the special case 1 and contradicts our assumptions. Therefore, {1, 2, 3, 6} cannot be irreducibly independent of x. In table 6, we list all four-term sets that could lead to new solutions based on table 5, the requirements of their complementary one- or two-term sets, and whether the four-term set is compatible with these requirements.

Table 6 shows that there are only 4 four-term sets that cannot be eliminated in this manner. We now investigate each of these possibilities. We begin by examining the possibility that {1, 2, 5, 6} is irreducibly independent of x. In this case, $X_1 X_2 - \frac{\mathrm{d}X_2}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}a} X_2 + \left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right) X_1$ must be independent of x. We do not know *a priori* the functional form of $u(a)$ or $\frac{\mathrm{d}u}{\mathrm{d}a}$, but we do know that $\frac{\mathrm{d}u}{\mathrm{d}a}$ must depend on a or this would reduce to a special case. Since terms 1 and 2 are independent of a, this condition requires that the x-dependence of $\frac{\mathrm{d}u}{\mathrm{d}a} X_2$ be canceled by the x-dependence of $\left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right) X_1$. For this to be true, the a-dependence of $\frac{\mathrm{d}u}{\mathrm{d}a}$ must differ from the a-dependence of $u + a\frac{\mathrm{d}u}{\mathrm{d}a}$ by only a multiplicative constant, in which case $X_2 + \alpha X_1 = \beta$ for some constants α and β. By shifting the zero of a, we can absorb X_2 into X_1, so this particular case can be reduced to the special case of $X_2 = 0$ and does not yield any new solutions.

The four-term set {1, 3, 5, 6} could be irreducibly independent of x if single-term sets {2} and {4} are each separately independent of x. However, from table 2, this would require that both X_1 and X_2 be linear in x. Therefore, $X_2 + \alpha X_1 = \beta$ for some constants α and β, which once again reduces to the special case $X_2 = 0$.

Table 6. For a four-term set to be irreducibly independent of x, either the two remaining terms must each be separately independent of x or the combination of the two remaining terms must produce an irreducibly independent two-term set. The first column of this table lists all possible four-term sets not eliminated from consideration by table 5. The second column shows possible two-term sets or pairs of single-term sets that could combine with the four-term set to produce an x-independent solution. The third column lists whether the four-term set is compatible with the requirements of the one- and two-term sets given by tables 2 and 3. If it is not, the final column lists the requirement that is incompatible with the four-term set. If all requirements are compatible, the final column is blank.

Four-term set	Complementary two-term set or the pair of one-term sets	Is the four-term set compatible?	Requirement incompatible with the four-term set
$\{1, 2, 3, 5\}$	$\{4\}, \{6\}$	No	$u = \alpha/a$
$\{1, 2, 3, 5\}$	$\{4, 6\}$	No	$u = \frac{\alpha}{a} + \gamma$
$\{1, 2, 3, 6\}$	$\{4, 5\}$	No	$u = \frac{\alpha a^2}{2} + \gamma$
$\{1, 2, 4, 6\}$	$\{3, 5\}$	No	$u = \frac{\alpha a^2}{2} + \gamma$
$\{1, 2, 5, 6\}$	$\{3, 4\}$	Yes	
$\{1, 3, 4, 5\}$	$\{2\}, \{6\}$	No	$u = \alpha/a$
$\{1, 3, 4, 5\}$	$\{2, 6\}$	No	$u = \frac{\alpha}{a} + \gamma$
$\{1, 3, 5, 6\}$	$\{2\}, \{4\}$	Yes	
$\{2, 3, 4, 5\}$	$\{1\}, \{6\}$	No	$u = \alpha/a$
$\{2, 3, 4, 5\}$	$\{1, 6\}$	No	$u = \frac{\alpha}{a} + \gamma$
$\{2, 3, 5, 6\}$	$\{1\}, \{4\}$	Yes	
$\{3, 4, 5, 6\}$	$\{1\}, \{2\}$	Yes	
$\{3, 4, 5, 6\}$	$\{1, 2\}$	Yes	

Finally, the four-term sets $\{2, 3, 5, 6\}$ and $\{3, 4, 5, 6\}$ can be analyzed using the same method as for the full six-term set. For example, the four-term set $\{2, 3, 5, 6\}$ can be irreducibly independent of x only if $-\frac{\mathrm{d}X_2}{\mathrm{d}x} + aX_1^2 + \frac{\mathrm{d}u}{\mathrm{d}a}X_2 + \left(u + a\frac{\mathrm{d}u}{\mathrm{d}a}\right)X_1$ does not depend on x. We do not know *a priori* the functional form of u, so we once again define functions $f_1(a)$ and $f_2(a)$ such that $u + a\frac{\mathrm{d}u}{\mathrm{d}a} = \alpha_1 + \beta_1 a + f_1(a)$ and $\frac{\mathrm{d}u}{\mathrm{d}a} = \alpha_2 + \beta_2 a + f_2(a)$, where $f_1(a)$ and $f_2(a)$ contain no constant terms or terms linear in a. With this definition, $-\frac{\mathrm{d}X_2}{\mathrm{d}x} + aX_1^2 + (\alpha_2 + a\beta_2 + f_2(a))X_2 + (\alpha_1 + a\beta_1 + f_1(a))X_1$ is independent of x. Since $f_1(a)$ and $f_2(a)$ contain no constant terms or terms proportional to a, the x dependence of the term $f_1(a)X_1$ cannot be canceled by terms 1–3, and neither can the x-dependence of $f_2(a)X_2$. Therefore, we conclude that $f_1(a)X_1 + f_2(a)X_2$ cannot depend on x. This requires either that $f_1(a) = f_2(a) = 0$ (which is reducible as discussed for the six-term solution) or the x-dependence of $f_1(a)X_1$ must be canceled by the x-dependence of $f_2(a)X_2$. This is possible only if $X_2 = \alpha X_1 + \beta$. The same argument obtains for $\{3, 4, 5, 6\}$. Therefore, both of these four-term sets can be reduced to the special case of $X_2 = 0$ and do not yield any new solutions.

By examining all possible four-term sets, we have found that no new solutions are admitted by any combinations that are not included as one of our special cases. Therefore, we consider the final possibility of solutions containing five-term sets. For a five-term set to be irreducibly independent of x, the remaining single-term set must be independent of x as well. As single-term sets $\{3\}$ and $\{5\}$ are incompatible with new solutions (cf table 2), this immediately leaves only four remaining possibilities. We examine each in turn.

The set $\{1, 2, 3, 4, 5\}$ can only be irreducibly independent of x if the single-term set $\{6\}$ is independent of x. From table 2, this requires that $u = \alpha/a$. Substituting this result into the remaining terms in (16) requires that $X_1X_2 - \frac{\mathrm{d}X_2}{\mathrm{d}x} + aX_1^2 - a\frac{\mathrm{d}X_1}{\mathrm{d}x} - (\alpha/a^2)X_2$ be independent of x. Since there is only one term proportional to a^{-2}, this can only be true if $\alpha = 0$ (implying that $u = 0$) or if $X_2 = 0$. Either possibility reduces to a special case.

The remaining five-term sets can be analyzed using the same method we used when considering the full six-term set. For example, the five-term set $\{1, 2, 3, 5, 6\}$ can be irreducibly independent of x only if $X_1X_2 - \frac{dX_2}{dx} + aX_1^2 + \frac{du}{da}X_2 + \left(u + a\frac{du}{da}\right)X_1$ does not depend on x. We once again define functions $f_1(a)$ and $f_2(a)$ such that $u + a\frac{du}{da} = \alpha_1 + \beta_1 a + f_1(a)$ and $\frac{du}{da} = \alpha_2 + \beta_2 a + f_2(a)$, where $f_1(a)$ and $f_2(a)$ contain no constant terms or terms linear in a. With this definition, $X_1X_2 - \frac{dX_2}{dx} + aX_1^2 + (\alpha_2 + a\beta_2 + f_2(a))X_2 + (\alpha_1 + a\beta_1 + f_1(a))X_1$ is independent of x. Since $f_1(a)$ and $f_2(a)$ contain no constant terms or terms proportional to a, we again conclude that $f_1(a)X_1 + f_2(a)X_2$ is independent of x. This requires either that $f_1(a) = f_2(a) = 0$, or $X_2 = \alpha X_1 + \beta$, both of which are reducible.

This same argument holds for the remaining two five-term sets: $\{1, 3, 4, 5, 6\}$ and $\{2, 3, 4, 5, 6\}$. Thus, we have examined all possible combinations of one-, two-, three-, four-, five-, and six-term sets and discovered that they allow no solutions other than those covered by one of the special cases. Since the special cases include all known conventional potentials, we therefore conclude that this method finds all of such known potentials, and it precludes other solutions. We have thus proven that the set of known $\hbar$-independent solutions is complete.

4. Superpotentials that contain explicit $\hbar$ dependence

Thus far, we have found all known additive shape-invariant superpotentials that do not depend explicitly on $\hbar$ and have proven that no more can exist. We now show that our formalism can be generalized to include 'extended' superpotentials that contain $\hbar$ explicitly. In this case, we expand the superpotential W in powers of $\hbar$:

$$W(x, a, \hbar) = \sum_{n=0}^{\infty} \hbar^n W_n(x, a). \tag{17}$$

We wish to substitute (17) in (7). From (17),

$$\left.\frac{\partial W}{\partial x}\right|_{a=a_0} = \sum_{n=0}^{\infty} \hbar^n \frac{\partial W_n(x, a_0)}{\partial x},$$

and

$$W^2(x, a_0, \hbar) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \hbar^{k+l} W_k W_l.$$

Since $a_1 = a_0 + \hbar$, $W(x, a_1, \hbar) = W(x, a_0 + \hbar, \hbar)$. Expanding in powers of $\hbar$,

$$W(x, a_1, \hbar) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{\hbar^m}{k!} \left.\frac{\partial^k W_{m-k}}{\partial a^k}\right|_{a=a_0}.$$

So

$$W^2(x, a_1, \hbar) = \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{k=0}^{s} \frac{\hbar^n}{(n-s)!} \frac{\partial^{n-2}}{\partial a^{n-2}} (W_k W_{s-k}).$$

Similarly,

$$\left.\frac{\partial W}{\partial x}\right|_{a=a_1} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{\hbar^m}{k!} \frac{\partial^{k+1} W_{m-k}}{\partial a^k \partial x}.$$

After significant algebraic manipulation and requiring that the result must hold for any value of $\hbar$, we find that the following equation must be true separately for each positive integer value of n:

$$\sum_{k=0}^{n} W_k\, W_{n-k} + \frac{\partial W_{n-1}}{\partial x} - \sum_{s=0}^{n}\sum_{k=0}^{s} \frac{1}{(n-s)!}\frac{\partial^{n-s}}{\partial a^{n-s}} W_k\, W_{s-k} + \sum_{k=1}^{n} \frac{1}{(k-1)!}\frac{\partial^{k}}{\partial a^{k-1}\,\partial x}\, W_{n-k} - \left(\frac{1}{n!}\,\frac{\partial^n g}{\partial a^n}\right) = 0. \tag{18}$$

For $n = 1$, we obtain

$$2\frac{\partial W_0}{\partial x} - \frac{\partial}{\partial a}\left(W_0^2 + g\right) = 0. \tag{19}$$

We have shown that all conventional superpotentials $W = W_0$ are solutions of this equation. The extended cases [13, 15] are solutions to (18) as well, as shown in [16]. It should be noted that in references [13, 15], the authors chose to set $\hbar = 1$, however, we do not. In addition, new potentials can be generated by applying (18) for all $n > 1$, as we show in the next section.

5. A new $\hbar$-dependent superpotential

5.1. Generating a new $\hbar$-dependent superpotential

We now use the method outlined in the previous section to generate a new superpotential. We begin by choosing a conventional shape-invariant solution to satisfy (19): $W_0 = (\alpha - a)\tanh x - B\,\operatorname{sech} x$, which is a Scarf II superpotential from table 1 with $A = (\alpha - a) > 0$. For $n = 2$, the expansion in (18) yields

$$\frac{\partial W_1}{\partial x} - \frac{\partial}{\partial a}\,(W_0 W_1) = 0,$$

and for $n = 3$, we obtain

$$\frac{\partial W_2}{\partial x} - \frac{\partial\left(2W_0W_2 + W_1^2\right)}{\partial a} - \frac{1}{2}\,\frac{\partial^2 W_0 W_1}{\partial a^2} + \frac{2}{3}\,\frac{\partial^3 W_0}{\partial a^2 \partial x} = 0.$$

Without imposing boundary conditions, there are many solutions to (18) for each value of n. However, from physical considerations we require that (1) solutions should not have singularities worse that $1/x^2$ to prevent domain splitting or particles being sucked into the singularity [38–40]; and (2) the asymptotic limits of W be the same as those for the corresponding W_0, so that supersymmetry remains unbroken.

With these considerations, these two coupled equations are solved by $W_1 = 0$ and $W_2 = -\frac{B\cosh x}{(a-\alpha-B\sinh x)^2}$. The next order equations are solved by $W_3 = 0$ and $W_4 = -\frac{B\cosh x}{4(a-\alpha-B\sinh x)^4}$. Generalizing these, for $n \geqslant 1$ we find

$$W_{2n-1} = 0; \qquad W_{2n} = -\frac{4B\cosh x}{(2a - 2\alpha - 2B\sinh x)^{2n}},$$

yielding a sum that converges to

$$W(x, a, \hbar) = (\alpha - a)\tanh x - B\,\operatorname{sech} x - 2B\hbar\cosh x\left(\frac{1}{2(a - \alpha - B\sinh x) - \hbar} - \frac{1}{2(a - \alpha - B\sinh x) + \hbar}\right). \tag{20}$$

This is a hitherto undiscovered superpotential that meets the requirements of shape invariance.

5.2. Analyzing the new $\hbar$-dependent superpotential

5.2.1. Partner potentials and shape invariance. A list of the associated potential V_-, eigenvalues E_n^-, and eigenfunctions $\psi_n^{(-)}$ corresponding to each conventional superpotential can be found in [10, 11]. Once these are known, the partner potential V_+ can be calculated, along with the corresponding E_n^+ and $\psi_n^{(+)}$. We now proceed to do the same for our new superpotential found in (20). To begin with, recall that this superpotential is simply an $\hbar$-dependent extension of a Scarf II superpotential. For the conventional Scarf II superpotential, the potentials $V_\pm$ are given by

$$\begin{aligned} W_0^2(x,a) \pm \hbar \frac{\mathrm{d}W_0(x,a)}{\mathrm{d}x} &= ((\alpha - a)\tanh x - B\,\mathrm{sech}\,x)^2 \\ &\quad \pm (\alpha - a)\hbar\,\mathrm{sech}^2 x \pm B\hbar\,\mathrm{sech}\,x\tanh x. \end{aligned} \tag{21}$$

For the new superpotential found in (20), we can similarly determine the associated partner potentials. For notational convenience, we define $\eta \equiv 1/[2(a - \alpha - B\sinh x) - \hbar]$ and $\xi \equiv 1/[2(a - \alpha - B\sinh x) + \hbar]$. With this notation, we find

$$\begin{aligned} V_\pm = W^2(x,a) \pm \hbar \frac{\mathrm{d}W(x,a)}{\mathrm{d}x} &= ((\alpha - a)\tanh x - B\,\mathrm{sech}\,x + 2B\hbar\,(\eta - \xi)\cosh x)^2 \\ &\quad \pm (\alpha - a)\,\hbar\,\mathrm{sech}^2 x \pm B\,\hbar\,\mathrm{sech} x\tanh x \pm 4B^2\hbar^2(\eta^2 - \xi^2)\cosh^2 x\,. \end{aligned} \tag{22}$$

To verify that these partner potentials indeed meet the requirements of shape invariance, we calculate

$$V_+(x,a) - V_-(x,a+\hbar) = W^2(x,a) + \hbar\frac{\mathrm{d}W(x,a)}{\mathrm{d}x} - W^2(x,a+\hbar) + \hbar\frac{\mathrm{d}W(x,a+\hbar)}{\mathrm{d}x}.$$

After significant algebra, most terms cancel, leaving

$$V_+(x,a) - V_-(x,a+\hbar) = \hbar\,(2\alpha - 2a - \hbar)\,. \tag{23}$$

This satisfies the shape invariance condition (4), where

$$\begin{aligned} g(a+\hbar) - g(a) &= \hbar(2\alpha - 2a - \hbar) \\ &= (\alpha - a)^2 - (\alpha - a - \hbar)^2. \end{aligned} \tag{24}$$

Thus, $g(a) = -(\alpha - a)^2$ to within an additive constant.

5.2.2. Eigenvalues of the new $\hbar$-dependent superpotential. One of the major advantages of working with shape-invariant superpotentials is that once we know the function $g(a)$, we know all eigenvalues. For extended superpotentials, from (19), we find that the function $g(a)$ is given by

$$\begin{aligned} &\frac{\partial g}{\partial a} = 2\frac{\partial W_0}{\partial x} - \frac{\partial W_0^2}{\partial a}. \text{ This yields} \\ &g(a) = 2\int \mathrm{d}a\,\left(\frac{\partial W_0}{\partial x}\right) - W_0^2 + f(x), \end{aligned} \tag{25}$$

where $f(x)$ is an arbitrary function of x with no a-dependence that ensures that (25) yields a function $g(a)$ of a alone. Thus, the function $g(a)$ is given entirely in terms of the $\hbar$-independent part of the superpotential. Hence, the eigenvalues are not affected by the $\hbar$-dependent extension of the superpotential.

In the case of the extended superpotential given in (20), $g(a) = -(\alpha - a)^2$ yields

$$E_n^- = g(a + n\hbar) - g(a) = (\alpha - a)^2 - (\alpha - a - n\hbar)^2 = A^2 - (A - n\hbar)^2 \tag{26}$$

which is identical to the energy spectrum for the Scarf II potential.

5.2.3. Eigenfunctions of the new $\hbar$-dependent superpotential. The ground state eigenfunction is given by $\psi_0^{(-)}(x, a_0) \sim \exp\left(\int_{x_0}^x W(x, a_0) \mathrm{d}x\right)$. We solve this integral by splitting W into its $\hbar$-dependent and -independent parts, so that $W_0 = (\alpha - a)\tanh x - B\,\mathrm{sech}\, x$ is the conventional Scarf II superpotential, and $W_\hbar = -2B\hbar\cosh x\left(\frac{1}{2(a-\alpha-B\sinh x)-\hbar} - \frac{1}{2(a-\alpha-B\sinh x)+\hbar}\right)$ is the $\hbar$-dependent part. Therefore, $\psi_0^{(-)}(x, a) \sim \exp\left(\int_{x_0}^x W(x, a)\,\mathrm{d}x\right)$ is given by

$$\begin{aligned}\psi_0^{(-)}(x, a) &\sim \exp\left(\int_{x_0}^x W_0(x, a)\,\mathrm{d}x\right)\exp\left(\int_{x_0}^x W_\hbar(x, a)\,\mathrm{d}x\right)\\ &\sim (\mathrm{sech}\, x)^{a-\alpha}(\mathrm{sech}\, x + \tanh x)^{-B}\left(\frac{2a - 2\alpha - \hbar - 2B\sinh x}{2a - 2\alpha + \hbar - 2B\sinh x}\right)^\hbar\\ &\sim (\mathrm{sech}\, x)^{-A}(\mathrm{sech}\, x + \tanh x)^{-B}\left(\frac{-2A - \hbar - 2B\sinh x}{-2A + \hbar - 2B\sinh x}\right)^\hbar.\end{aligned} \tag{27}$$

Higher excited states can be found by repeatedly applying A^+ to the ground state:

$$\psi_n^{(-)}(x, a) \sim \Pi_{k=0}^{k=n-1}\left(-\hbar\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a + k\hbar)\right)\psi_0^{(-)}(x, a + n\hbar). \tag{28}$$

6. Conclusions

We have transformed the condition for additive shape invariance into a set of local partial differential equations. For conventional cases that do not depend on $\hbar$, we have shown that the shape invariance condition is equivalent to an Euler equation expressing momentum conservation for fluids and an equation of constraint. Solving these equations we have generated all known $\hbar$-independent shape-invariant superpotentials, and we have also shown that there are no others.

For extended cases in which the superpotential depends explicitly on $\hbar$, we have developed equation (18) that is satisfied by all additive shape-invariant superpotentials. We have developed a recursive method to solve this equation, and used it to show that the energy spectrum of the extended system is identical to its conventional counterpart. Earlier [16] we used this method to reproduce the results found by Quesne [13]; in this paper, we have generated a new shape-invariant superpotential and have examined its properties.

This method may be exploited to obtain other $\hbar$-dependent superpotentials by starting with each of the $\hbar$-independent ones. We hope to develop an algorithm which can do so, as well as a test for completeness for the family of shape-invariant superpotentials thus generated. Thus, this method thus has the possibility of greatly expanding our ability to identify shape-invariant superpotentials.

It may also be possible to extend this method to other forms of shape invariance such as multiplicative or cyclic. For these types of shape invariance, the potentials are generally not available in terms of known functions, except in very special cases ($N = 2$ for cyclic and limiting cases for multiplicative). It remains to be shown whether for these, the shape invariance condition can be transformed from a difference-differential equation into a set of partial differential equations and be subjected to similar analysis.

Acknowledgment

This research was supported by an award from Research Corporation for Science Advancement.

References

[1] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[2] Bhattacharjie A and Sudarshan E C G 1962 *Nuovo Cimento* **25** 864
[3] Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 219
[4] Ginocchio J N 1984 *Ann. Phys., NY* **152** 203
[5] Cooper F, Ginocchio J and Khare A 1987 *Phys. Rev.* D **36** 2458
[6] Levai G 1991 *J. Phys. A: Math. Gen.* **24** 131
[7] Witten E 1982 *Nucl. Phys.* B **202** 253
[8] Cooper F and Freedman B 1983 *Ann. Phys.* **146** 262
[9] Junker G 1992 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[10] Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)
[11] For a recent review of SUSYQM see Gangopadhyaya A, Mallow J V and Rasinariu C 2010 *Supersymmetric Quantum Mechanics: An Introduction* (Singapore: World Scientific)
[12] Fock V 1928 *Z. Phys.* **49** 339
[13] Quesne C 2008 *J. Phys. A: Math. Theor.* **41** 392001
[14] Odake S and Sasaki R 2009 *Phys. Lett.* B **679** 414
[15] Odake S and Sasaki R 2010 *J. Math. Phys.* **51** 053513
[16] Bougie J, Gangopadhyaya A and Mallow J V 2010 *Phys. Rev. Lett.* **105** 210402
[17] Gendenshtein L E 1983 *JETP Lett.* **38** 356
[18] For a recent review, see Cariñena J F and Ramos A 2000 *Rev. Math. Phys.* **12** 1279
[19] Balantekin A B 1998 *Phys. Rev.* A **57** 4188
[20] Gangopadhyaya A, Mallow J V and Sukhatme U P 1997 *Proc. Workshop on Supersymmetry and Integrable Models* ed H Aratyn *et al* (Berlin: Springer) pp 341–50
[21] Gangopadhyaya A, Mallow J V and Sukhatme U P 1998 *Phys. Rev.* A **58** 4287
[22] Barclay D, Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U P 1993 *Phys. Rev.* A **48** 2786
[23] Sukhatme U P, Rasinariu C and Khare A 1997 *Phys. Lett.* A **234** 401
[24] Spiridonov V P 1992 *Phys. Rev. Lett.* **69** 398
[25] Khare A and Sukhatme U P 1993 *J. Phys. A: Math. Gen.* **26** L901
[26] Dutt R, Khare A and Sukhatme U P 1998 *Am. J. Phys.* **56** 163
[27] Gangopadhyaya A and Mallow J V 2008 *Int. J. Mod. Phys.* A **23** 4959
[28] Levai G 1989 *J. Phys. A: Math. Gen.* **22** 689
[29] Cariñena J F and Ramos 2000 *J. Phys. A: Math. Gen.* **33** 3467
[30] Shabat A 1992 *Inverse Problems* **8** 303
[31] Curtright T and Fairlie D 2003 *J. Phys. A: Math. Gen.* **36** 8885 and references therein
[32] Kwong W and Rosner J 1986 *Prog. Theor. Phys. Suppl.* **86** 366
[33] Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2297
[34] Sukumar C V 1987 *J. Phys. A: Math. Gen.* **20** 2461
[35] Kwong W, Riggs H, Rosner J and Thacker H B 1989 *Phys. Rev.* D **39** 1242
[36] Wang Q, Sukhatme U, Keung W-Y and Imbo T 1990 *Mod. Phys. Lett.* A **5** 525
[37] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* 2nd edn (New York: Pergamon) p 3
[38] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (New York: Pergamon)
[39] Frank W M, Land D J and Spector R M 1971 *Rev. Mod. Phys.* **43** 36
[40] Gangopadhyaya A, Panigrahi P K and Sukhatme U P 1994 *J. Phys. A: Math. Gen.* **27** 4295