International Journal of Modern Physics A
Vol. 23, No. 31 (2008) 4959–4978
© World Scientific Publishing Company

# GENERATING SHAPE INVARIANT POTENTIALS

ASIM GANGOPADHYAYA* and JEFFRY V. MALLOW†
*Department of Physics, Loyola University Chicago, Chicago, Illinois 60626, USA*
* *agangop@luc.edu*
* *asim@uic.edu*
† *jmallow@luc.edu*

Received 26 April 2008
Revised 17 July 2008

We transform the shape invariance condition, a difference-differential equation of supersymmetric quantum mechanics, into a local partial differential equation. We develop a new method for generating translationally shape invariant potentials from this equation. We generate precisely all the known shape invariant potentials, and argue that there are unlikely to be others.

*Keywords*: Supersymmetric quantum mechanics; shape invariance; difference-differential equations; exactly solvable potentials.

## 1. Introduction

Supersymmetric quantum mechanics (SUSYQM).[1–4] is one of the very few ways to obtain the energy spectrum of quantum mechanical potentials without solving the Schrödinger equation.[5–9] It extends the factorization method for the harmonic oscillator to a large number of other potentials, whose energy eigenvalues can then be obtained algebraically. The SUSYQM extension consists of introducing a superpotential $W(x,a)$ that generates two partner Hamiltonians, both with the same energy eigenvalues.[a] These partner Hamiltonians are given by

$$H_{\mp} = A^{\pm}A^{\mp} = \left(\mp\hbar\frac{d}{dx} + W(x,a)\right)\left(\pm\hbar\frac{d}{dx} + W(x,a)\right)$$
$$= -\hbar^2\frac{d^2}{dx^2} + W^2(x,a) \mp \hbar\frac{dW(x,a)}{dx} = -\hbar^2\frac{d^2}{dx^2} + V_{\mp}(x,a)\,, \tag{1}$$

where partner potentials $V_{\mp}(x,a)$ are related to superpotential $W$ by $W^2(x,a) \mp \hbar\frac{dW(x,a)}{dx}$. Knowing the spectrum of one partner immediately gives information about the spectrum of the other.

[a]One of the partner potentials has an "extra" zero-energy ground state for *the* unbroken SUSY case.

However, for any pair of partner potentials, one would still have to know the spectrum of one of the partners to obtain the other. The critical advance in SUSYQM was the discovery[10] and subsequent rediscovery of shape invariance (SI).[11,12] Originally, finding shape invariant superpotentials was left to excellent intuition or trial and error. In 1987, Cooper *et al.*[13] developed a systematic method for generating the known translationally shape invariant superpotentials, using a parametrized $W(a,x)$. In another work[14] we argued, based on group theoretical considerations, that all of the known cases of translationally shape invariance are the only ones which can exist.

A system is called shape invariant if partner potentials could be shown to differ only by the value of a parameter and an additive constant:

$$V_+(x,a_0) + g(a_0) = V_-(x,a_1) + g(a_1)\,,$$

in which case each spectrum can be generated without reference to its partner. Parameter $a_1$ is a function of $a_0$; i.e. $a_1 = f(a_0)$. Energy eigenvalues of $H_-(x,a_0)$ are then given by $E_n = g(f^n(a_0)) - g(a_0)$, where $f^n(a_0)$ means that function $f$ was applied $n$ times to $(a_0)$. When expressed in terms of the superpotential, the shape invariance condition is

$$W^2(x,a_0) + \hbar\,\frac{dW(x,a_0)}{dx} + g(a_0) = W^2(x,a_1) - \hbar\,\frac{dW(x,a_1)}{dx} + g(a_1)\,. \qquad (2)$$

This condition is a difference-differential equation, and has also been studied for dynamical systems, where it is known as the infinite-dimensional dressing.[15]

In Sec. 2 we shall derive the partial differential equation equivalent to this difference-differential equation. As far as we know, this is the first time the shape invariance condition has been expressed as a local condition on the two-dimensional $(x,a)$ plane. We demonstrate two examples of its application: the Morse and Scarf I potentials.

In Sec. 3 we shall introduce a general ansatz for obtaining solutions to the equation: an $N$-term expansion of the superpotential. To our knowledge, this approach had also never been used in SUSYQM.

In Sec. 4 we shall apply the ansatz to the cases of $N=1$ and $N=2$ and show that these can produce all of the known translationally shape invariant potentials, and that the known list is complete to this order.

In the appendix we prove the linear dependence of the coefficients in the ansatz results. This is a necessary condition for obtaining the solutions of the partial differential equation.

## 2. The Partial Differential Equation

Most of the known exactly solvable superpotentials are such that their parameters differ by an additive constant; i.e. $a_1 = a_0 + \hbar$. This form of shape invariance is called "additive" or "translational."[1–4] This will be the case for all the shape invariant superpotentials we shall consider in this paper. (There are other forms of shape invariance such as multiplicative,[16] cyclic,[17] and more involved ones,[18,19] but we shall not consider them here.) We emphasize that the shape-invariance condition (Eq. (2)) is nonlinear in $W$; therefore one cannot simply add arbitrary constants to $W$ and maintain shape-invariance. This extremely strong constraint restricts the known shape invariant superpotentials to a very small number.

Equation (2) is a difference-differential equation; that is, it relates the square of the superpotential $W$ and its spatial derivative computed at two different parameter values: $a_0 \equiv a - \hbar$ and $a_1 \equiv a$. Our objective is to show that this integrability condition can be expressed as a nonlinear partial differential equation that is local in nature; i.e. all terms in the equation can be computed at the same point $(x, a)$. This has the obvious advantage of mathematical familiarity (at least to physicists). It provides a systematic method for finding the superpotential, and thus $g(a_n)$ and from it the energy spectrum of translationally shape invariant superpotentials. We shall show that all known translationally shape invariant superpotentials[1–4] are its solutions.

Note that we have made no assumptions about $\hbar$: for now, it is simply a part of the parameter defining $W$. Since this equation is valid for any value of $\hbar$, we will treat $x$ and $a - \hbar$ on equal footing as independent variables. Using the above definitions of $a_0$ and $a_1$, Eq. (2) can now be written as

$$W^2(x, a-\hbar) + \hbar\,\frac{\partial W(x, a-\hbar)}{\partial x} + g(a-\hbar) = W^2(x, a) - \hbar\,\frac{\partial W(x, a)}{\partial x} + g(a)\,. \quad (3)$$

If we now differentiate Eq. (3) with respect to $\hbar$, noting that $W^2(x, a)$ and $g(a)$ are independent of $\hbar$, we obtain

$$\begin{aligned} &2W(x, a-\hbar)\,\frac{\partial W(x, a-\hbar)}{\partial \hbar} + \frac{\partial W(x, a-\hbar)}{\partial x} \\ &\quad + \hbar\,\frac{\partial}{\partial \hbar}\,\frac{\partial W(x, a-\hbar)}{\partial x} + \frac{\partial g(a-\hbar)}{\partial \hbar} = -\frac{\partial W(x, a)}{\partial x}\,. \end{aligned} \quad (4)$$

Now because of translational shape invariance, the dependence of $W$ on $a$ and $\hbar$ is through the linear combination $a - \hbar$; therefore, the derivatives of $W$ with respect to $a$ and $\hbar$ are related by

$$\frac{\partial W(x, a-\hbar)}{\partial \hbar} = -\frac{\partial W(x, a-\hbar)}{\partial a}\,.$$

Note that at this point we depart from the standard approach of treating $x$ as a continuous variable and $a$ as a discrete parameter. Since the value of $a$ is unspecified, we treat them both on equal footing as continuous variables. Then we may rewrite Eq. (4):

$$\begin{aligned} &-2W(x, a-\hbar)\frac{\partial W(x, a-\hbar)}{\partial a} + \frac{\partial W(x, a-\hbar)}{\partial x} \\ &\quad - \hbar\,\frac{\partial}{\partial a}\,\frac{\partial W(x, a-\hbar)}{\partial x} - \frac{\partial g(a-\hbar)}{\partial a} = -\frac{\partial W(x, a)}{\partial x}\,. \end{aligned}$$

But this must be true for any value of $\hbar$. If we then set $\hbar = 0$, shape invariance implies the following condition for the superpotential $W(x, a)$:

$$2W(x,a)\left(\frac{\partial W(x,a)}{\partial a}\right) - 2\,\frac{\partial W(x,a)}{\partial x} + \frac{\partial g(a)}{\partial a} = 0\,. \tag{5}$$

This is the desired partial differential equation which necessarily follows from the original difference-differential equation for shape invariance. Thus, we have shown that all shape invariant superpotentials with $a_1 = a_0 + \hbar$ are solutions of the above nonlinear partial differential equation. Before we develop a method to determine solutions of this equation; i.e. to find shape invariant superpotentials, we will give two examples of known shape invariant superpotentials, show that they are solutions of this differential equation, and obtain their spectra.

**Morse potential:** The superpotential is given by

$$W(x, A) = A - Be^{-x}\,, \tag{6}$$

where $A$ is the parameter that changes additively. Substituting this in Eq. (5), we get: $2A + \frac{\partial g(A)}{\partial A} = 0$, whence $A = -\frac{1}{2}\,\frac{\partial g(A)}{\partial A}$, which gives $g(A) = -A^2$.

Setting $a_0 = -A$, we get $a_n \equiv f^n(a_0) = a_0 + n\hbar = -A + n\hbar$. Then $E_n = g(a_n) - g(a_0) = A^2 - (A - n\hbar)^2$.

**Scarf I potential:** The superpotential is given by

$$W(x, A) = A\tan x + B\sec x\,, \tag{7}$$

where $A$ is the parameter that changes. Substituting this superpotential into Eq. (5), we get

$$2(A\tan x + B\sec x)(\tan x) - 2(A\sec^2 x + B\sec x\tan x) + \frac{\partial g(A)}{\partial A} = 0\,,$$

i.e. $\frac{\partial g(A)}{\partial A} - 2A = 0$, hence $g(A) = A^2$. Thus, setting $a_0 = A$, $a_n = a_0 + n\hbar = A + n\hbar$, and $E_n = g(a_n) - g(a_0) = (A + n\hbar)^2 - A^2$.

One can check that all examples listed in Refs. 1–4 and 20 satisfy the partial differential equation given in Eq. (5). For a given shape invariant superpotential, Eq. (5) determines $g(a)$, and hence its eigenspectrum can be obtained.

However, our goal here is the opposite. We develop a novel method for generating these superpotentials. The remainder of this paper shows how this can be done.

Since all translationally shape invariant superpotentials are solutions to Eq. (5), finding solutions to this equation will help us in our quest for shape invariant superpotentials. Our method, a variant of the standard separation of variables, generates solutions of this nonlinear partial differential equation. Using this method, we are able to derive all the known translationally shape invariant superpotentials.

## 3. Ansatz

At this point, it is tempting to solve Eq. (5) by assuming various forms for the term $\frac{\partial g(a)}{\partial a}$. However, instead of that rather random process, we choose to try a factorizable ansatz:

$$W(x,a) = \sum_{i=1}^{N} \mathcal{A}_i(a)\mathcal{X}_i(x)\,, \tag{8}$$

where $\mathcal{A}_i(a)$ and $\mathcal{X}_i(x)$ are assumed to be respectively real functions of $a$ and $x$. Thus, to determine $W(x,a)$, we will need to find $2N$ functions $\mathcal{A}_i(a)$ and $\mathcal{X}_i(x)$. Equation (5) provides one equation involving these functions. So we need another $2N-1$ constraints to fully determine $W(x,a)$. We will consider the above ansatz for $N=1$ and $N=2$. Throughout this work, we shall use the following notation:

- lower case Greek letters will denote "true" ($a$- and $x$-independent) constants;
- upper case Latin letters with $a$ in parentheses will denote $a$-dependent, but $x$-independent "constants."

## 4. Generation of the Shape Invariant Potentials

$N=1$: *Single-term superpotentials*

$$W(x,a) = \mathcal{A}(a)\mathcal{X}(x)\,.$$

Substituting this ansatz into Eq. (5) leads to

$$2(\mathcal{A}(a)\mathcal{X}(x))\left(\mathcal{X}(x)\frac{d\mathcal{A}(a)}{da}\right) - 2\mathcal{A}(a)\frac{d\mathcal{X}(x)}{dx} + \frac{dg(a)}{da} = 0$$

$$\Rightarrow 2\mathcal{A}\underbrace{\left(\mathcal{X}^2\frac{d\mathcal{A}}{da} - \frac{d\mathcal{X}}{dx}\right)}_{H(a)} + \frac{dg}{da} = 0\,. \tag{9}$$

In Eq. (9), we see that the expression $\left(\mathcal{X}^2\frac{d\mathcal{A}}{da} - \frac{d\mathcal{X}}{dx}\right)$ can at most be a function of the parameter $a$, and hence, we set it equal to $H(a)$. For $H(a) \neq 0$,[b] this leads to

$$\mathcal{X} = -\sqrt{\frac{H}{\frac{d\mathcal{A}}{da}}}\tanh\left(\sqrt{H\,\frac{d\mathcal{A}}{da}}(x-x_0)\right). \tag{10}$$

Since, $\mathcal{X}(x)$ is independent of $a$, we must have, $H(a) \sim \frac{d\mathcal{A}(a)}{da}$ and $H(a) \sim \left(\frac{d\mathcal{A}(a)}{da}\right)^{-1}$. These two conditions can only be met if $H(a)$ and $\frac{d\mathcal{A}(a)}{da}$ are constants, independent of the parameter $a$. Thus $\mathcal{A}$ must be a linear function of $a$: $\mathcal{A} = \mu a + \beta$, and $H(a) = \gamma$ where the Greek-letter constants are all $a$-independent.

[b]For $H(a) = 0$, the solution to the above differential equation is $\mathcal{X} \sim 1/x$ which is a special case of the Coulomb potential with angular momentum $l = 0$. Since we will consider the general case later, we do not discuss it here.

This leads to the superpotential $W(x,a) = -(\mu a + \beta)\sqrt{\frac{\gamma}{\mu}}\tanh(\sqrt{\gamma\mu}(x-x_0))$. By a scaling ($\gamma\mu = 1$) and a shift ($x_0 = 0$) of $x$, we can write

$$W(x,a) = A\tanh x\,, \tag{11}$$

where $A = \left(\frac{\beta}{\mu} - a\right)$. This superpotential represents unbroken SUSY and holds a finite number of bound states. It is a special case of the Scarf II and the Rosen–Morse superpotentials.[1–4,20]

Note that for negative values of $H$, instead of the hyperbolic trigonometric function we get the superpotential $W(x,a) = A\tan x$. Note also that this formulation includes terms like cot and coth as solutions to the same differential equation for different values of $H(a)$ and $\frac{d\mathcal{A}}{da}$. For $\mathcal{A} = \text{const}$, Eq. (9) generates

$$W(x,a) = \frac{\omega}{2}x\,, \tag{12}$$

the superpotential for the *harmonic oscillator*, where we have set $H(a) = -\frac{1}{2}\omega$.

This exhausts the solutions for the $N = 1$ ansatz. Note that by definition it can only give one of the terms in any superpotential. Thus the only complete shape invariant superpotential we find is the harmonic oscillator. Therefore, we now set $N = 2$. This will allow us to obtain the whole family of two-term superpotentials.

$N = 2$: *Two-term superpotentials*

$$W(x,a) = \sum_{i=1}^{2}\mathcal{A}_i(a)\mathcal{X}_i(x) = \mathcal{A}_1(a)\mathcal{X}_1(x) + \mathcal{A}_2(a)\mathcal{X}_2(x)\,.$$

*We would like to very strongly emphasize that neither the functions $\mathcal{A}_1(a)$ and $\mathcal{A}_2(a)$, nor $\mathcal{X}_1(x)$ and $\mathcal{X}_2(x)$, should be linearly proportional to each other, otherwise this case will reduce to the case of $N = 1$ considered above.* Substituting the above ansatz in Eq. (5) leads to

$$2(\mathcal{A}_1\mathcal{X}_1 + \mathcal{A}_2\mathcal{X}_2)\left(\mathcal{X}_1\frac{d\mathcal{A}_1}{da} + \mathcal{X}_2\frac{d\mathcal{A}_2}{da}\right) + \frac{dg(a)}{da} = \left(\mathcal{A}_1\frac{d\mathcal{X}_1}{dx} + \mathcal{A}_2\frac{d\mathcal{X}_2}{dx}\right).$$

Expanding it, we get

$$\underbrace{2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2}_{\text{Term \#1}}\ \underbrace{-\,2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx}}_{\text{Term \#2}}\ \underbrace{+\,2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2}_{\text{Term \#3}}\ \underbrace{-\,2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx}}_{\text{Term \#4}}$$
$$\underbrace{+\,2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)}_{\text{Term \#5}} + \underbrace{\frac{dg(a)}{da}}_{\text{Term \#6}} = 0\,. \tag{13}$$

This is of the form

$$\sum_{i}^{6}F_i(a)G_i(x) = 0\,,$$

where

$$
\begin{aligned}
F_1 &\equiv 2\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}\,; & G_1 &\equiv \mathcal{X}_1^2\,, \\
F_2 &\equiv -2\mathcal{A}_1\,; & G_2 &\equiv \frac{d\mathcal{X}_1}{dx}\,, \\
F_3 &\equiv 2\mathcal{A}_2 \frac{d\mathcal{A}_2}{da}\,; & G_3 &\equiv \mathcal{X}_2^2\,, \\
F_4 &\equiv -2\mathcal{A}_2\,; & G_4 &\equiv \frac{d\mathcal{X}_2}{dx}\,, \\
F_5 &\equiv 2\left(\mathcal{A}_1 \frac{d\mathcal{A}_2}{da} + \mathcal{A}_2 \frac{d\mathcal{A}_1}{da}\right); & G_5 &\equiv \mathcal{X}_1\mathcal{X}_2\,, \\
F_6 &\equiv \frac{dg}{da}\,; & G_6 &\equiv 1\,.
\end{aligned}
\tag{14}
$$

This linear combination of various functions of $x$ must yield us functions $\mathcal{X}_1$ and $\mathcal{X}_2$ that are independent of $a$. This severely constrains the $\mathcal{A}_i$'s and $\mathcal{X}_i$'s. In the appendix we prove the following theorem. *If a combination of $F_iG_i$ is irreducibly constant*; *i.e. no smaller combination is independent of $x$, then all $F_i$'s must be proportional to each other.* We will be making extensive use of this property throughout the paper.

In the notation we will be using, we will refer to $F_iG_i$ as the $i$th term; e.g. $2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2$ as term #1, $2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx}$ as term #2, etc. as explicitly shown in Eq. (13). Of these terms, the sum of the first five terms is $x$-independent because the sixth term, $\frac{dg(a)}{da}$, does not depend on $x$. However, a subcollection of the first five terms may also add up to a constant. If, for example, terms #2, #3 and #4 add up to a constant and no smaller group of them is a constant, we call such a group irreducibly constant and denote it by $\{2, 3, 4\}$.

Our general method will be driven by the following constraints:

- Any term or set of terms that is irreducible must be at most only $a$-dependent, since the sum of all the terms is $-dg(a)/da$.
- Since each term is a mixture of functions of $\mathcal{A}_i$ and $\mathcal{X}_i, i = 1, 2$, the former may have $a$-dependence, but the latter must not.
- In any irreducibly constant subgroup, the functions of $\mathcal{A}_i$ (the $F_i$'s above) must be proportional to each other, as proved in the appendix.
- For those cases where only a single term is irreducible, then either the function of $\mathcal{X}_i$ must be a constant or the function of $\mathcal{A}_i$ must be zero, to preserve the dependence on $a$ alone.

(i) Let us first consider the case that the sum $\sum_{i=1}^{5} F_i(a)G_i(x)$ is irreducible; that is, no smaller group of terms is constant. We will denote this group by $\{1, 2, 3, 4, 5\}$. In this case all $F_i$ are proportional to each other. The proportionality of $F_2$ and $F_4$ implies that the $\mathcal{A}_i$'s are proportional and hence reduces to the $N = 1$ case.

(ii) Now let us consider the case where there are four-term combinations independent of $x$. This also implies that there is one $x$-independent term, which leads to severe constraint on $\mathcal{X}_i$'s. The remaining four-term group has all corresponding $F_i$'s proportional to each other.

There are five such possibilities, of which, due to symmetry between the first and third terms and between the second and fourth, two of the terms are similar to others. For example, the condition "$\mathcal{X}_1 = \text{const}$" is equivalent to "$\mathcal{X}_2 = \text{const}$."[c]

This leads to three cases:

(a) $\{2, 3, 4, 5\} + \{1\}$

$$-2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx} + 2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 - 2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx} + 2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) \text{ is an } a\text{-dependent constant}.$$

From the first and the third term of the above expression, irreducibility of the sum implies that $\mathcal{A}_1 \sim \mathcal{A}_2$; i.e. $\mathcal{A}_1$ and $\mathcal{A}_2$ are linearly proportional to each other, and hence the problem reduces to the $N = 1$ case. We would like to emphasize this feature. *Whenever an irreducible group contains both term #2 and term #4, we have $\mathcal{A}_1 \sim \mathcal{A}_2$, and hence a case of $N = 1$.*

(b) $\{1, 3, 4, 5\} + \{2\}$

$$2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 + 2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 - 2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx} + 2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) \equiv H_1(a) \tag{15}$$

and $\left[2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx}\right] = -\frac{dg(a)}{da} - H_1(a)$. From the irreducibility of $\{1, 3, 4, 5\}$, we have

$$\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \mathcal{A}_2 \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) \sim H(a)\,.$$

From $\mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \mathcal{A}_2$, we get $\mathcal{A}_2 = \alpha a + \beta$. From $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_2\frac{d\mathcal{A}_2}{da}$, we get $(\mathcal{A}_1)^2 = (\mathcal{A}_2)^2 + \delta = (\alpha a + \beta)^2 + \delta$, where $\delta \neq 0$. However, from $\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) \sim \mathcal{A}_2 = (\alpha a + \beta)$, we get $\mathcal{A}_1 = \frac{\eta}{\alpha a+\beta} + \frac{\kappa}{2\alpha}(\alpha a + \beta)$. These two expressions for $\mathcal{A}_1$ are consistent only if $\delta = \eta = 0$, which implies $\mathcal{A}_1 \sim \mathcal{A}_2$, thus reducing to the $N = 1$ case.

(c) $\{1, 2, 3, 4\} + \{5\}$

$$2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - 2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx} + 2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 - 2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx} \equiv H(a)\,, \tag{16}$$

[c] As we did in the $N = 1$ case, we must distinguish between constants that depend on $a$, and those that are $a$- and $x$-independent constants.

and $\left[2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da}+\mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)\right]=-\frac{dg(a)}{da}-H(a)$. From the second and fourth terms of Eq. (16), the irreducibility of $\{1,2,3,4\}$ implies that $\mathcal{A}_1\sim\mathcal{A}_2$. Thus this reduces to the $N=1$ case as well.

(iii) There are ten possible ways to set three-term groups equal to a constant. However, due to the symmetry of terms with respect to $\mathcal{X}_1$ and $\mathcal{X}_2$, we see that there are only six combinations we need to consider. We list them all below.

(a) Terms $\{1,2,3\}$ add up to an $a$-dependent constant. Let us first consider $\{1,2,3\}+\{4,5\}$.

$$2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2-2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx}+2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2=H(a)\,. \tag{17}$$

Since we have $\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\sim\mathcal{A}_1\sim\mathcal{A}_2\frac{d\mathcal{A}_2}{da}$, we find from the first condition that $\mathcal{A}_1$ must be a linear function of $a$, i.e. $\mathcal{A}_1=\alpha a+\beta$, and from the second condition that $(\mathcal{A}_2)^2=\gamma(\mathcal{A}_1)^2+\delta=\gamma(\alpha a+\beta)^2+\delta$. From the constancy of $\{4,5\}$, we have

$$-2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx}+2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da}+\mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)=J(a)\,. \tag{18}$$

This gives $\mathcal{A}_2\sim\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da}+\mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)\equiv\kappa\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da}+\mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$. Substituting $\mathcal{A}_1=(\alpha a+\beta)$, we get the following differential equation: $\kappa(\alpha a+\beta)\frac{d\mathcal{A}_2}{da}+(\kappa\alpha-1)\mathcal{A}_2=0$, which is solved by $\mathcal{A}_2=(\alpha a+\beta)^{\frac{1}{\kappa\alpha}-1}$. Thus we have two expressions for $\mathcal{A}_2$, both of which must be valid for all values of $a$. Their compatibility implies $\frac{1}{\kappa\alpha}=2$, $\gamma=1$ and $\delta=0$. Then $\mathcal{A}_2\sim\mathcal{A}_1$. This then is an $N=1$ case.

Now let us look at the case of $\{1,2,3\}+\{4\}+\{5\}$.

From, $\{4\}$, we get $\mathcal{X}_2=\alpha_2 x+\beta_2$. Since the fifth term is constant, we have two choices: it is either equal to zero or it is not.

- Let us first consider the case that the fifth term is zero. As in the previous case, from $\{1,2,3\}$ we have $\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\sim\mathcal{A}_1\sim\mathcal{A}_2\frac{d\mathcal{A}_2}{da}$; i.e. $\mathcal{A}_1=\alpha a+\beta$, and that $(\mathcal{A}_2)^2=\gamma(\mathcal{A}_1)^2+\delta$. But since the fifth term is zero, we also have $\mathcal{A}_1\sim\mathcal{A}_2^{-1}$. These two conditions can only be met if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants independent of $a$, and the problem reduces to the $N=1$ case.
- Now let us assume the fifth term is a nonzero $a$-dependent constant. Then $\mathcal{X}_1\sim 1/\mathcal{X}_2$. We have $\{1,2,3\}$ and $\{4\}$ separately equal to $a$-dependent constants. There are two possibilities for the fourth term: it is either zero or a nonzero constant. In the first instance, $\frac{d\mathcal{X}_2}{dx}=0$, that makes $\mathcal{X}_2$ a constant. This implies that $\{1,2,3\}$ is reducible, and hence will be considered later. However, if the fourth term is not zero then $\mathcal{X}_2=\alpha x+\beta$. This implies that $\mathcal{X}_1=\frac{\gamma}{\alpha x+\beta}$. Then the irreducible

three-term $\{1,2,3\}$ implies:

$$2\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}\left(\frac{\gamma}{\alpha x+\beta}\right)^2 - 2\mathcal{A}_1 \frac{\gamma\alpha}{(\alpha x+\beta)^2} + 2\mathcal{A}_2 \frac{d\mathcal{A}_2}{da}(\alpha x+\beta)^2 = M(a)\,.$$

This algebraic equation has a nontrivial solution for all values of $x$ only if $2\mathcal{A}_2\frac{d\mathcal{A}_2}{da} = 0$. However, that reduces $\{1,2,3\}$ to $\{1,2\}+\{3\}$ and hence will be considered later.

(b) Terms $\{1,2,4\}$ add up to an $a$-dependent constant. That is

$$2\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - 2\mathcal{A}_1 \frac{d\mathcal{X}_1}{dx} - 2\mathcal{A}_2 \frac{d\mathcal{X}_2}{dx} = H(a)\,.$$

Then from the irreducibility of $\{1,2,4\}$, $\mathcal{A}_1 \sim \mathcal{A}_2$: an $N=1$ case.

(c) Terms $\{1,2,5\}$ add up to an $a$-dependent constant.

$$2\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - 2\mathcal{A}_1 \frac{d\mathcal{X}_1}{dx} + 2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) = H(a)\,. \quad (19)$$

The remaining two terms could add up to a constant irreducibly or separately. Thus there are two possibilities for this case: $\{1,2,5\}+\{3,4\}$ or $\{1,2,5\}+\{3\}+\{4\}$. Before we analyze these cases, we note that from Eq. (19), we get $\mathcal{A}_1 \sim \mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$. Thus we have $\mathcal{A}_1 = \alpha_1 a + \beta_1$.

Now, we first consider the case of $\{1,2,5\}+\{3,4\}$. From $\{3,4\}$, we have

$$\mathcal{A}_2 \frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 - \mathcal{A}_2 \frac{d\mathcal{X}_2}{dx} = C(a)\,. \quad (20)$$

From Eq. (20), we get $\mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \mathcal{A}_2$, which gives $\mathcal{A}_2 = \alpha_2 a+\beta_2$. Substituting expressions for $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{A}_1 \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) = \alpha_2\mathcal{A}_1 + \alpha_1\mathcal{A}_2$ implies that $\mathcal{A}_2 \sim \mathcal{A}_1$, thus reducing this to an $N=1$ case.

Next we consider the case of $\{1,2,5\}+\{3\}+\{4\}$. From $\{1,2,5\}$, we still have $\mathcal{A}_1 = \alpha_1 a+\beta_1$ and $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) \sim \mathcal{A}_1 \equiv \gamma_1\mathcal{A}_1$. From $\{3\}$ there are two possibilities. Either $\mathcal{A}_2$ is constant or $\mathcal{X}_2$ is constant. If $\mathcal{A}_2$ is constant, from $\{1,2,5\}$ we find that $\mathcal{A}_1$ is also constant, and hence we have an $N=1$ case. Now we consider the case $\mathcal{X}_2 = \gamma_2$ is constant. The irreducible combination now leads to the differential equation

$$\alpha_1\mathcal{X}_1^2 - \frac{d\mathcal{X}_1}{dx} + \underbrace{\gamma_1\gamma_2}_{\gamma}\mathcal{X}_1 = \frac{H(a)}{2\mathcal{A}_1}\,,$$

whose solution is $\left(\tanh x - \frac{\gamma}{2\alpha_1}\right)$. To determine $\mathcal{A}_2$, we start with the equation $\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) = \gamma_1\mathcal{A}_1$; i.e.

$$(\alpha_1 a+\beta_1)\frac{d\mathcal{A}_2}{da} + \alpha_1\mathcal{A}_2 = \gamma_1(\alpha_1 a+\beta_1)\,. \quad (21)$$

It is solved by $\mathcal{A}_2 = \frac{\gamma_1\left(\frac{1}{2}\alpha_1 a^2+\beta_1 a\right)+\zeta_1}{\alpha_1 a+\beta_1}$. Solving for the superpotential $W(x,a) = \mathcal{A}_1\mathcal{X}_1 + \mathcal{A}_2\mathcal{X}_2$, we get

$$W(x,a) = (\alpha_1 a + \beta_1)\tanh x + \frac{\zeta_1 - \frac{\gamma\beta_1^2}{2\alpha_1}}{\alpha_1 a + \beta_1} = A\tanh x + \frac{B}{A}\,.$$

This is the *Rosen-Morse II* potential, with $A = \alpha_1 a + \beta_1$ and $B = \zeta_1 - \frac{\gamma\beta_1^2}{2\alpha_1}$.

(d) Terms $\{1,3,4\}$ add up to an $a$-dependent constant. This is the same as item (a).

(e) Terms $\{1,3,5\}$ add up to an $a$-dependent constant.

$$2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 + 2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 + 2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) = H(a)\,. \quad (22)$$

Equation (22) implies that we have $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$. $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_2\frac{d\mathcal{A}_2}{da}$ gives $(\mathcal{A}_1)^2 = \alpha_1(\mathcal{A}_2)^2 + \beta_1$. The proportionality condition $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$ generates $(\mathcal{A}_1)^2 = \alpha_2(\mathcal{A}_1\mathcal{A}_2) + \beta_2$. If $\beta_1$ or $\beta_2$ vanishes, we get $\mathcal{A}_1 \sim \mathcal{A}_2$. If $\beta_1$ and $\beta_2$ are different from zero, these two algebraic equations for $\mathcal{A}_1$ and $\mathcal{A}_2$ determine them to be constants. Thus, in both situations, we have an $N = 1$ case.

(f) Terms $\{1,4,5\}$ add up to an $a$-dependent constant.

Let us first consider the case $\{1,4,5\} + \{2,3\}$. Thus we have

$$2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - 2\mathcal{A}_2\frac{d\mathcal{X}_2}{dx} + 2\mathcal{X}_1\mathcal{X}_2\left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right) = H_1(a) \quad (23)$$

and

$$2\mathcal{A}_2\frac{d\mathcal{A}_2}{da}\mathcal{X}_2^2 - 2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx} = H_2(a)\,. \quad (24)$$

Equations (23) and (24) imply $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_2 \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$, and $\mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \mathcal{A}_1$ respectively. Thus we have $\frac{d\mathcal{A}_1}{da} \sim \frac{\mathcal{A}_2}{\mathcal{A}_1}$ and $\frac{d\mathcal{A}_2}{da} \sim \frac{\mathcal{A}_1}{\mathcal{A}_2}$. From $\mathcal{A}_1\frac{d\mathcal{A}_1}{da} \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$, we get $\mathcal{A}_2 = \delta_1\mathcal{A}_1 + \frac{\delta_2}{\mathcal{A}_1}$. Substituting above expressions for $\frac{d\mathcal{A}_1}{da}$, $\frac{d\mathcal{A}_2}{da}$, and $\mathcal{A}_2$ into $\mathcal{A}_2 \sim \left(\mathcal{A}_1\frac{d\mathcal{A}_2}{da} + \mathcal{A}_2\frac{d\mathcal{A}_1}{da}\right)$, we get the following algebraic equation for $\mathcal{A}_1$:

$$\delta_1\mathcal{A}_1 + \frac{\delta_2}{\mathcal{A}_1} = \frac{\kappa_1\mathcal{A}_1^2}{\delta_1\mathcal{A}_1 + \frac{\delta_2}{\mathcal{A}_1}} + \frac{\left(\delta_1\mathcal{A}_1 + \frac{\delta_2}{\mathcal{A}_1}\right)^2}{\mathcal{A}_1}\,,$$

whose solution $\mathcal{A}_1$ equals a constant. However, this then makes $\{1,4,5\}$ irreducible, hence will be considered later.

Now consider the case $\{1,4,5\} + \{2\} + \{3\}$. From $\{3\}$, we have $\mathcal{A}_2$ a constant because $\mathcal{X}_2$ constant makes $\{1,4,5\}$ reducible. Using this in $\{1,4,5\}$, we get $\mathcal{A}_1^2 = \gamma_1 a + \delta_1$ and $\mathcal{A}_1^2 = \gamma_2\mathcal{A}_1\mathcal{A}_2 + \delta_2$. Since $\mathcal{A}_2$ is constant, from the second condition we get $\mathcal{A}_1$ constant. The problem thus reduces to the $N = 1$ case.

(g) Terms $\{2,3,4\}$ add up to an $a$-dependent constant. This is the same as (b).
(h) Terms $\{2,3,5\}$ add up to an $a$-dependent constant. This is the same as (f).
(i) Terms $\{2,4,5\}$ add up to an $a$-dependent constant. As stated earlier, the presence of term #2 and term #4 in $\{2,4,5\}$ implies $\mathcal{A}_1 \sim \mathcal{A}_2$ and the problem is immediately reduced to an $N=1$ case.
(j) Terms $\{3,4,5\}$ add up to an $a$-dependent constant. This is the same as (c).

(iv) As we set out to now consider all irreducible two term cases, let us recognize again that term #1 in Eq. (13) is equivalent to term #3 and term #2 is equivalent to term #4. There are several possibilities.

(a) • $\{1,2\}+\{3,4\}+\{5\}$

$\{1,2\}+\{3,4\}+(\{5\}=0)$

The first and second terms add to an $a$-dependent constant: $2\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - 2\mathcal{A}_1\frac{d\mathcal{X}_1}{dx} = H(a)$. The proportionality of the coefficients on the left-hand side imply $\frac{d\mathcal{A}_1}{da}=\alpha$. Then $\mathcal{A}_1 = \alpha a+\eta$. Similarly, the constancy of the combination of the third and fourth terms gives $\mathcal{A}_2 = \alpha_2 a + \eta_2$.

But from $\{5\}$, $\mathcal{A}_2 \sim 1/\mathcal{A}_1$; i.e. $\alpha_2 a+\eta_2 \sim 1/(\alpha_1 a+\eta_1)$. For this to be true for all values of $a$, $\alpha_1=\alpha_2=0$. Then both $\mathcal{A}_1$ and $\mathcal{A}_2$ are $a$-independent constants: a special case of $\mathcal{A}_2 \sim \mathcal{A}_1$, and thus an $N=1$ case.

$\{1,2\}+\{3,4\}+(\{5\}\neq 0)$

In this case, instead of $\mathcal{A}_2 \sim 1/\mathcal{A}_1$, from $\{5\}$ we have $\mathcal{X}_2 \sim 1/\mathcal{X}_1$. The differential equations from $\{1,2\}$ and $\{3,4\}$ give respectively $\mathcal{X}_1 = \tanh x$ and $\mathcal{X}_2 = \coth x$. As a result, the superpotential for this case will be of the form

$$W(x,a) = (\alpha_1 a+\eta_1)\tanh(\gamma x) + (\alpha_2 a+\eta_2)\coth(\gamma x)\,. \qquad (25)$$

With the identification of $(\alpha_1 a+\eta_1)=A$ and $(\alpha_2 a+\eta_2)=-B$, we get $A\tanh r - B\coth r$. This is the *Pöschl-Teller II* superpotential. This also produces, with different constants, the *Pöschl-Teller I* superpotential.[20]

• $\{1,2\}+\{3,5\}+\{4\}$

The constancy of the fourth term implies that $\frac{d\mathcal{X}_2}{dx}=\alpha$; i.e. $\mathcal{X}_2=\alpha x+\beta$. We cannot consider the case of $\alpha=0$ here because that would make $\mathcal{X}_2$ constant, and thus $\{3,5\}$ would be reducible to $\{3\}$ and $\{5\}$. We will consider this later. From the irreducibility of $\{3,5\}$, we obtain

$$\mathcal{A}_2\frac{d\mathcal{A}_2}{da}(\mathcal{X}_2)^2 + \mathcal{X}_1\mathcal{X}_2\frac{d}{da}(\mathcal{A}_1\mathcal{A}_2) = D(a)\,. \qquad (26)$$

From this we get $\mathcal{A}_2\frac{d\mathcal{A}_2}{da} \sim \frac{d}{da}(\mathcal{A}_1\mathcal{A}_2)$, i.e. $\mathcal{A}_1 = \theta_1\mathcal{A}_2 + \frac{\theta_2}{\mathcal{A}_2}$. This leads to

$$(\mathcal{X}_2)^2 + \theta_1\mathcal{X}_1\mathcal{X}_2 = \kappa\,. \qquad (27)$$

For $\kappa=0$, we have an algebraic equation which reduces to $\mathcal{X}_1 \sim \mathcal{X}_2$; thus an $N=1$ case.

For $\kappa \neq 0$, from $\{1,2\}$, we have $\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}(\mathcal{X}_1)^2 - \mathcal{A}_1 \frac{d\mathcal{X}_1}{dx} = J(a)$, i.e. $\mathcal{A}_1 \frac{d\mathcal{A}_1}{da} \sim \mathcal{A}_1$. This gives $\mathcal{A}_1 = \gamma a + \nu$, which yields

$$\gamma(\mathcal{X}_1)^2 - \frac{d\mathcal{X}_1}{dx} = \lambda\,.$$

For $\lambda = 0$, we get $\mathcal{X}_1 = -\frac{\gamma}{x - x_0}$. This is not compatible with the condition given in Eq. (27) and $\mathcal{X}_2 = \alpha x + \beta$. Thus there is no solution.
For $\lambda \neq 0$, we get two cases again. $\gamma$ could be equal to zero or a nonzero constant. For $\gamma = 0$, $\{1,2\}$ becomes reducible, and hence will be pursued later. For $\gamma \neq 0$, the solution for $\mathcal{X}_1$ will be a function of the type tan, cot, tanh or coth, none of which is compatible with $\mathcal{X}_2 = \alpha x + \beta$ and Eq. (27). Thus there is no solution.

- $\{1,2\} + \{4,5\} + \{3\}$

  Constancy of the term $\{3\}$ implies that either $\mathcal{X}_2 = \xi$, or that the $\frac{d\mathcal{A}_2}{da} = 0$; i.e. $\mathcal{A}_2 = \alpha_2$.

— If $\mathcal{X}_2$ is a constant, then $\{4,5\}$ becomes reducible and will be discussed later.

— If $\mathcal{A}_2$ is constant, from $\{4,5\}$ we have $\mathcal{A}_1 = \alpha a + \beta$. From $\{1,2\}$, we have

$$\alpha \mathcal{X}_1^2 - \frac{d\mathcal{X}_1}{dx} = \lambda\,. \tag{28}$$

For $\lambda = 0$, we get $\mathcal{X}_1 = -\frac{1}{\alpha x - \zeta_1}$. Substituting in the equation (from $\{4,5\}$)

$$\frac{d\mathcal{X}_2}{dx} - \alpha \mathcal{X}_1 \mathcal{X}_2 = \nu\,, \tag{29}$$

and solving for $\mathcal{X}_2$, we get

$$\mathcal{X}_2 = \frac{\nu\left(\frac{\alpha}{2}x^2 - \zeta_1 x\right) + \zeta_2}{\alpha x - \zeta_1}\,.$$

The superpotential is then given by

$$\frac{\alpha a + \beta + \frac{\nu\xi\zeta_1}{2\alpha} + \zeta_2}{\alpha x - \zeta_1} - \frac{\xi\nu}{2\alpha}(\alpha x - \zeta_1)\,,$$

which for $\alpha a + \beta + \frac{\nu\xi\zeta_1}{2\alpha} + \zeta_2 = l + 1$, $-\frac{\xi\nu}{2\alpha} = \frac{\omega}{2}$ and $\alpha x - \zeta_1 = r$, becomes

$$W(r,a) = \frac{1}{2}\omega r - \frac{l+1}{r}\,,$$

the *three-dimensional harmonic oscillator.*
For $\lambda \neq 0$, $\mathcal{X}_1$ is of the form $\tan x$. Then solving for $\mathcal{X}_2$ in Eq. (29), we get $\mathcal{X}_2$ of the form $\sec x$. Hence, the superpotential generated is of the form

$$W(x,A) = A\tan x + B\sec x\,.$$

Depending on values of constants $\alpha, \nu$ and $\lambda$, Eqs. (28) and (29) also produce the shape invariant superpotential

$$W(x,a) = A\coth x - B\,\mathrm{cosech}\,x\,.$$

This is the *generalized Pöschl-Teller* superpotential.[1–4] Similarly, Eqs. (28) and (29) can also be used to generate the *Scarf I* ($A\tan x - B\sec x$) and *Scarf II* ($A\tanh x + B\,\mathrm{sech}\,x$) superpotentials.

- $\{1,2\}+\{3\}+\{4\}+\{5\}$
  From {1,2}, we have $\mathcal{A}_1\frac{d\mathcal{A}_1}{da}\mathcal{X}_1^2 - \mathcal{A}_1\frac{d\mathcal{X}_1}{dx} = M(a)$. This gives $\frac{d\mathcal{A}_1}{da} = \alpha$, constant; thus, $\mathcal{A}_1 = \alpha a + \beta$. For $M(a) \neq 0$, we then get $\mathcal{X}_1 \sim$ tanh or $\mathcal{X}_1 \sim$ coth or their trigonometric counterparts; i.e. tan and cot. For $M(a) = 0$, the solution is $\mathcal{X}_1 = -\frac{\alpha}{x}$.
  For $\mathcal{X}_2$, let us turn to the remaining terms. The fifth term is either zero or a nonzero constant. We will consider them separately.
  $\{1,2\}+\{3\}+\{4\}+(\{5\}=0)$
  From {3}, either $\mathcal{X}_2$ or $\mathcal{A}_2$ is a constant. If $\mathcal{X}_2$ is constant, with $\mathcal{A}_1 \sim 1/\mathcal{A}_2$ from $\{5\} = 0$ and $M(a) \neq 0$, we generate the following superpotentials: *Rosen Morse I* $\left(-\frac{B}{A} - A\cot x\right)$, *Rosen Morse II* $\left(\frac{B}{A} + A\tanh x\right)$, *Eckart* $\left(\frac{B}{A} - A\coth x\right)$. For $M(a) = 0$, we get instead $W(x,a) = -\frac{\alpha(\alpha a+\beta)}{x} + \frac{\gamma}{\alpha a+\beta}$. With the identifications $x \to r$, $\alpha(\alpha a+\beta) \to (l+1)$ and $\frac{\gamma}{(\alpha a+\beta)} = \frac{e^2}{2(l+1)}$, we obtain the *Coulomb* superpotential.
  $\{1,2\}+\{3\}+\{4\}+(\{5\}\neq 0)$
  From $(\{5\}\neq 0)$, we have $\mathcal{X}_1 \sim 1/\mathcal{X}_2$. Thus, $\mathcal{X}_2 =$ const is no longer a solution. From {3} and {4}, we find that $\mathcal{X}_2$ must be a linear function of $x$ and $\mathcal{A}_2$ must be a constant. Thus the only solution for $\mathcal{X}_1$ that is compatible with $\mathcal{X}_1 \sim 1/\mathcal{X}_2$ and linearity of $\mathcal{X}_2$ is $\mathcal{X}_1 = -\frac{\alpha}{x}$. The superpotential generated is $W(x,a) = -\frac{\alpha(\alpha a+\beta)}{x} + \gamma x$. With the identification of $x \to r$, $\alpha(\alpha a+\beta) \to (l+1)$, and $\gamma \to \frac{1}{2}\omega$, we get (again) the superpotential of the *three-dimensional harmonic oscillator*:

$$W(r,a) = \frac{1}{2}\omega r - \frac{l+1}{r}\,,$$

  the same as $\{1,2\}+\{4,5\}+\{3\}$.

(b)
- $\{1,3\}+\{2,4\}+\{5\}$
  From $\{2,4\}$, this combination immediately reduces to an $N=1$ case.
- $\{1,3\}+\{2\}+\{4\}+\{5\}$
  Let us first consider $\{1,3\}+\{2\}+\{4\}+(\{5\}=0)$.
  Since {2} and {4} are constants, we have $\mathcal{X}_1 = \alpha_1 x + \beta_1$ and $\mathcal{X}_2 = \alpha_2 x + \beta_2$. From $\{1,3\}$, we have

$$(\alpha_1 x+\beta_1)^2 + \kappa(\alpha_2 x+\beta_2)^2 = \delta\,.$$

The above equation is valid for all values of $x$ only if $\alpha_1 x + \beta_1 = \sqrt{-\kappa}(\alpha_2 x + \beta_2)$; i.e. $\mathcal{X}_1 \sim \mathcal{X}_2$. Thus we have a case of $N = 1$.

This satisfies $\{1,3\}$ only if $\mathcal{X}_1$ and $\mathcal{X}_2$ are proportional. However, that again reduces the problem to the $N = 1$ case.

$\{1,3\} + \{2\} + \{4\} + (\{5\} \neq 0)$
In this case since $\frac{d\mathcal{X}_1}{dx}$, $\frac{d\mathcal{X}_2}{dx}$ and $\mathcal{X}_1\mathcal{X}_2$ are all constant, we get the trivial case of constant functions $\mathcal{X}_1$ and $\mathcal{X}_2$.

- $\{1,3\} + \{2,5\} + \{4\}$
From $\{4\}$, we have $\mathcal{X}_2 = \gamma x + \delta$. Since terms 1 and 3 add irreducibly to a constant, we have $(\mathcal{X}_1)^2 + \kappa(\mathcal{X}_2)^2 = \nu$. From this, we get $\mathcal{X}_1 = \sqrt{\nu - \kappa\,(\gamma x + \delta)^2}$. Substituting these expressions of $\mathcal{X}_1$ and $\mathcal{X}_2$ in the combination {2,5}, we see that it is satisfied for a range of value of $x$ only if $\gamma = 0$. Thus, we get the trivial solution of $\mathcal{X}_2$ and $\mathcal{X}_2$ both constant.
- $\{1,3\} + \{4,5\} + \{2\}$ is equivalent to the case $\{1,3\} + \{2,5\} + \{4\}$ considered above; viz., the trivial solution.
- $\{1,3\} + \{2\} + \{4\} + \{5\}$
First consider $\{1,3\} + \{2\} + \{4\} + (\{5\} = 0)$. Since the sum of the first and third terms, $2\mathcal{A}_1\mathcal{X}_1^2\frac{d\mathcal{A}_1}{da} + 2\mathcal{A}_2\mathcal{X}_2^2\frac{d\mathcal{A}_2}{da} = B(a)$, we have $\mathcal{A}_2^2 = \epsilon\mathcal{A}_1^2 + \gamma$. But from $(\{5\} = 0)$ we have $\mathcal{A}_2 \sim 1/\mathcal{A}_1$. Thus the only solution is that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are $a$-independent constants, hence the special case of $\mathcal{A}_2 \sim \mathcal{A}_1$; viz., the $N = 1$ case.

  Next we consider $\{1,3\} + \{2\} + \{4\} + (\{5\} \neq 0)$.

  From $(\{5\} \neq 0)$, we have $\mathcal{X}_2 \sim 1/\mathcal{X}_1$. Terms $\{2\}$ and $\{4\}$ imply that $\mathcal{X}_1$ and $\mathcal{X}_2$ are linear functions of $x$. Compatibility among $\{2\}$, $\{4\}$, and $\{5\}$ thus requires that $\mathcal{X}_1$ and $\mathcal{X}_2$ both be constants, therefore we get the trivial solution again.

(c) • $\{1,4\} + \{2,3\} + \{5\}$
Let us first consider $\{1,4\} + \{2,3\} + (\{5\} = 0)$. In this case, from $\{1,4\}$ we have $\frac{d(\mathcal{A}_1)^2}{da} \sim \mathcal{A}_2$ and from $(\{5\} = 0)$ we have $\mathcal{A}_1 \sim 1/\mathcal{A}_2$. The product of these two conditions give $\mathcal{A}_1\frac{d(\mathcal{A}_1)^2}{da} \sim$ const, i.e. $\frac{d(\mathcal{A}_1)^3}{da} = \alpha$, a constant. This gives $\mathcal{A}_1 = (\alpha a+\beta)^{1/3}$ and $\mathcal{A}_2 = (\alpha a+\beta)^{-1/3}$. However, from $\{2,3\}$, we also have $\frac{d(\mathcal{A}_2)^2}{da} \sim \mathcal{A}_1$. This last constraint implies that we must have $\alpha = 0$, i.e. $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants; viz. the $N = 1$ case.

Now let us consider $\{1,4\} + \{2,3\} + (\{5\} \neq 0)$.

From $(\{5\} \neq 0)$, we have $\mathcal{X}_1 = \gamma/\mathcal{X}_2$. From $\{2,3\}$ we get $\frac{d\mathcal{X}_1}{dx} + \alpha_1\mathcal{X}_2^2 = \beta_1$, and from $\{1,4\}$ we have $\frac{d\mathcal{X}_2}{dx} + \alpha_2\mathcal{X}_1^2 = \beta_2$. Differentiating $\mathcal{X}_1 = \gamma/\mathcal{X}_2$, we get $\frac{d\mathcal{X}_1}{dx} = -\gamma\frac{d\mathcal{X}_2}{dx}/(\mathcal{X}_2)^2$. Eliminating $\mathcal{X}_1$ and its derivative from these two equations, we get an algebraic equation for $\mathcal{X}_2$, whose solution is $\mathcal{X}_2 = \delta$. This, along with $\mathcal{X}_1 = \gamma/\mathcal{X}_2$, reduces the problem to a trivial case of $\mathcal{X}_1$ and $\mathcal{X}_2$ equal to constants.

- $\{1,4\}+\{2,5\}+\{3\}$
  Constancy of the term $\{3\}$ implies that either $\mathcal{X}_2$ is constant or $\frac{d(\mathcal{A}_2)^2}{da}=0$.

  — First consider the case that $\mathcal{X}_2$ is constant. This gives $\{4\}=0$; i.e. $\{1,4\}$ becomes a reducible case to be considered later.
  — For the other possibility, $\frac{d(\mathcal{A}_2)^2}{da}=0$, $\mathcal{A}_2$ is a constant. Then from $\{1,4\}$, we get $\frac{d}{da}(\mathcal{A}_1)^2=\theta_1$. Which gives $\mathcal{A}_1=\sqrt{\theta_1 a+\theta_2}$. However, from $\{2,5\}$ we get $\mathcal{A}_1\sim\frac{d}{da}\mathcal{A}_1$, i.e. $\mathcal{A}_1=\gamma_1 e^{\gamma_2 a}+\gamma_3$. These two expressions are not compatible unless both $\mathcal{A}_1$ and $\mathcal{A}_2$ are constants, in which case $N=1$.

- $\{1,4\}+\{3,5\}+\{2\}$
  $\{2\}$ implies that $\frac{d\mathcal{X}_1}{dx}=$ constant. So, $\mathcal{X}_1=\alpha_1 x+\beta_1$.

  — $\alpha_1=0$.
  Thus $\mathcal{X}_1=\beta_1$. This makes $\{1,4\}$ reducible.
  — $\alpha_1\neq 0$.
  From $\{1,4\}$, we have $\mathcal{X}_2\sim(\alpha_1 x+\beta_1)^3+\gamma$.
  But $\{3,5\}$ implies $\mathcal{X}_2^2\sim\mathcal{X}_1\mathcal{X}_2+\delta$. This expression is not compatible with the values of $\mathcal{X}_1$ and $\mathcal{X}_2$ above, unless $\alpha_1=0$; i.e. $\mathcal{X}_1$ and $\mathcal{X}_2$ are constant, the trivial case once again.

- $\{1,4\}+\{2\}+\{3\}+\{5\}$
  As we saw above, constancy of the term $\{3\}$ implies that either $\mathcal{X}_2$ is constant or $\mathcal{A}_2$ is constant, and $\mathcal{X}_2$ constant will not be considered, since it makes $\{1,4\}$ reducible. Thus $\mathcal{A}_2$ is constant. For $\{5\}=0$, $\mathcal{A}_1\sim 1/\mathcal{A}_2=$ a constant, thus reducing to an $N=1$ case. For $\{5\}\neq 0$, we have $\mathcal{X}_1\sim 1/\mathcal{X}_2$. Term $\{2\}$ constant means that $\mathcal{X}_1=\alpha_1 x+\beta_1$, hence $\mathcal{X}_2\sim 1/(\alpha_1 x+\beta_1)$. But from $\{1,4\}$, $\frac{d\mathcal{X}_2}{dx}=\epsilon\mathcal{X}_1^2+\delta$. So $\mathcal{X}_2=\epsilon(\alpha_1 x+\beta_1)^3/(3\alpha_1)+\delta x$. These are incompatible expressions; therefore, there is no solution.

(d)
- $\{1,5\}+\{2,3\}+\{4\}$. This is equivalent to $\{1,4\}+\{3,5\}+\{2\}$ analyzed earlier.
- $\{1,5\}+\{2,4\}+\{3\}$. The presence of $\{2,4\}$ immediately implies an $N=1$ case.
- $\{1,5\}+\{3,4\}+\{2\}$ is equivalent to $\{1,2\}+\{3,5\}+\{4\}$ analyzed earlier.
- $\{1,5\}+\{2\}+\{3\}+\{4\}$. From $\{2\}$ and $\{4\}$, we have $\mathcal{X}_1=\alpha_1 x+\beta_1$ and $\mathcal{X}_2=\alpha_2 x+\beta_2$. Then from $\{3\}$, we have two possibilities: either $\mathcal{X}_2$ is a constant or $\mathcal{A}_2$ is constant.

  — We first consider the case $\mathcal{A}_2$ constant. This implies, from $\{1,5\}$, $\mathcal{A}_1$ is constant as well. This is an $N=1$ case.

— Now we consider the case $\mathcal{X}_2 = \beta_2$, a constant. Since we have $\mathcal{X}_1 = \alpha_1 x + \beta_1$, from $\{1,5\}$ we get

$$\mathcal{A}_1 \frac{d\mathcal{A}_1}{da}(\alpha_1 x + \beta_1)^2 + \beta_2(\alpha_1 x + \beta_1)\frac{d\mathcal{A}_1\mathcal{A}_2}{da} = J(a)\,.$$

This equation can be satisfied for all values of $x$ only if $\alpha_1 = 0$, in which case $\mathcal{X}_1$ is also constant; viz. a trivial solution, or $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent of $a$, reducing this to an $N = 1$ case.

(e) • $\{2,4\} + \{1\} + \{3\} + \{5\}$. $\{2,4\}$ gives $\mathcal{A}_1 \sim \mathcal{A}_2$, an $N = 1$ case.

(f) • $\{2,5\} + \{1\} + \{3\} + \{4\}$

Since term $\{1\}$ is constant, we have two choices. It is either equal to zero or a nonzero constant.

- If it is nonzero, that would imply that $\mathcal{X}_1$ is a constant. This reduces $\{2,5\}$ to $\{2\} + \{5\}$ and will be considered later.
- On the other hand, if term $\{1\}$ is zero, we must have $\mathcal{A}_1$ a constant (since $\mathcal{X}_1 = 0$ is not acceptable). So we choose $\mathcal{A}_1 = \eta$. From $\{2,5\}$ we then get $\frac{d\mathcal{A}_2}{da} = \alpha$, so $\mathcal{A}_2 = \alpha a + \beta$. Further, from $\{3\}$ we have $\mathcal{X}_2 = \delta$, a constant, since $\mathcal{A}_2$ is not. This then gives $\left(\frac{d\mathcal{X}_1}{dx} - \alpha\delta\mathcal{X}_1\right) = \mu$. The solution to this equation is $\mathcal{X}_1 = \Lambda e^{(\alpha\delta x)} - \frac{\mu}{\alpha\delta}$. The resulting superpotential $W(x,a) = \mathcal{A}_1\mathcal{X}_1 + \mathcal{A}_2\mathcal{X}_2$ is

$$W(x,A) = A - Be^{-\nu x}\,, \tag{30}$$

where $A = \delta(\alpha a + \beta) - \frac{\eta\mu}{\alpha\delta}$, $B = -\eta\Lambda$, $\nu = -\alpha\delta$. For $\alpha\delta < 0$, this gives the *Morse* superpotential.
- $\{1\} + \{2\} + \{3\} + \{4\} + \{5\}$

Let us consider various possibilities.

($\{1\} = 0$) implies $\mathcal{A}_1$ is a constant. ($\{1\} \neq 0$) implies $\mathcal{X}_1$ is a constant. ($\{3\} = 0$) implies $\mathcal{A}_2$ is a constant. ($\{3\} \neq 0$) implies $\mathcal{X}_2$ is a constant. So, if both ($\{1\} = 0$) and ($\{3\} = 0$), we have $\mathcal{A}_1 \sim \mathcal{A}_2$, hence $N = 1$. Similarly, if both ($\{1\} \neq 0$) and ($\{3\} \neq 0$), we have $\mathcal{X}_1$ and $\mathcal{X}_2$ equal constants, a trivial case. Thus the only case that we need to consider is ($\{1\} \neq 0$) and ($\{3\} = 0$). This equivalent to ($\{1\} = 0$) and ($\{3\} \neq 0$). In this case, we have $\mathcal{A}_1 = \nu_1$ and $\mathcal{X}_2 = \beta_2$. Then $\{5\}$ yields $\nu_1\beta_2\mathcal{X}_1\frac{\mathcal{A}_2}{da}$. If ($\{5\} = 0$), we have $\mathcal{A}_2 = \nu_2$; i.e. $\mathcal{A}_1 \sim \mathcal{A}_2$, an $N = 1$ case. If ($\{5\} \neq 0$), $\mathcal{X}_1 = \beta_1$ and hence a trivial case where both $\mathcal{X}_1$ and $\mathcal{X}_2$ are constants.

This exhausts all the possibilities for the $N = 2$ ansatz.

We have now obtained all known translationally shape invariant superpotentials. We have found no new ones, using the $N = 1$ and $N = 2$ ansatz. Table 1 lists these superpotentials, as well as one of the combinations of irreducible terms from which they were obtained.[d]

[d]As we have seen, there are multiple combinations of irreducible terms which generate the same superpotential.

Table 1.

| Name | Superpotential | $N$ | Combination |
|---|---|---|---|
| Harmonic oscillator | $\frac{1}{2}\omega x$ | 1 | |
| Coulomb | $\frac{e^2}{2(l+1)} - \frac{l+1}{r}$ | 2 | $\{1,2\}+\{3\}+\{4\}+(\{5\}=0)$ |
| Three-dimensional harmonic oscillator | $\frac{1}{2}\omega r - \frac{l+1}{r}$ | 2 | $\{1,2\}+\{3\}+\{4\}+(\{5\}\neq 0)$ |
| Morse | $A - Be^{-x}$ | 2 | $\{2,5\}+\{1\}+\{3\}+\{4\}$ |
| Pöschl-Teller I | $A\tan r - B\cot r$ | 2 | $\{1,2\}+\{3,4\}+(\{5\}\neq 0)$ |
| Pöschl-Teller II | $A\tanh r - B\coth r$ | 2 | $\{1,2\}+\{3,4\}+(\{5\}\neq 0)$ |
| Rosen-Morse I | $-A\cot x - \frac{B}{A}$ | 2 | $\{1,2\}+\{3\}+\{4\}+(\{5\}=0)$ |
| Rosen-Morse II | $A\tanh x + \frac{B}{A}$ | 2 | $\{1,2\}+\{3\}+\{4\}+(\{5\}=0)$ |
| Eckart | $-A\coth x + \frac{B}{A}$ | 2 | $\{1,2\}+\{3\}+\{4\}+(\{5\}=0)$ |
| Scarf I | $A\tan x - B\sec x$ | 2 | $\{1,2\}+\{4,5\}+\{3\}$ |
| Scarf II | $A\tanh x + B\,\mathrm{sech}\,x$ | 2 | $\{1,2\}+\{4,5\}+\{3\}$ |
| Generalized Pöschl-Teller | $A\coth x - B\,\mathrm{cosech}\,x$ | 2 | $\{1,2\}+\{4,5\}+\{3\}$ |

## 5. Conclusions

We have developed an *ab initio* method for generating shape invariant superpotentials, by transforming the shape invariance equation into a nonlinear partial differential equation. We have constructed a variant of the separation of variables method to find solutions of this equation, and have found that the list of solutions generated includes all known shape invariant superpotentials, as given in Refs. 1–4 and 20.

Since we have now generated precisely the full family of known translationally shape invariant potentials by two independent methods: the one in this paper and the earlier one based on group theory, we believe that all the known translationally shape invariant potentials constitute the complete set.

In principle, the algorithm provided here for $N = 1$ and $N = 2$ can be automated to find solutions for higher values of $N$. However, the major promise of this new method lies not in its ability to produce all known translationally invariant shape invariant potentials. It lies rather in providing a paradigm for analyzing systems with shape invariance, whether they are translational, multiplicative or cyclic.[16,17] Currently, there are no examples of multiplicative shape invariant systems for which the potential is known in a closed form, except for the limiting cases for which they coincide with translationally shape invariant systems. Similarly, for cyclic cases, only the potentials of order two are known exactly. For cyclic potentials of order three and higher, we only know their asymptotic behavior at infinity. The method proposed in this paper can be extended to analyze these other situations and, hopefully, to generate new solvable systems to enlarge the very small list of quantum mechanical systems that can be solved exactly.

## Appendix. Linear Dependence of Coefficients $F_i$

Equation (13) shows that not all $F_i$ are linearly independent. In particular, it states that at most five $F_i$ can be linearly independent and the dimensionality of the space they span would be at most five. We will show that if an expression $\sum_i F_i G_i$ irreducibly adds up to a constant, the space spanned by $F_i$ will have the dimensionality of one; i.e. all $F_i$ must be proportional to each other.

Let us first consider an expression consisting of just two elements: $\{F_1 G_1, F_2 G_2\}$, that add up to a constant irreducibly; i.e. $F_1(a)G_1(x) + F_2(a)G_2(x) = H(a)$. The irreducibility for this case implies that $F_1(a)G_1(x)$ and $F_2(a)G_2(x)$ cannot be separately constant. This implies that neither $G_1(x)$ nor $G_2(x)$ can be $x$-independent constants, and $F_1(a)$ and $F_2(a)$ cannot be equal to zero, otherwise the two terms would be reducible to one.

Dividing the above equality by $H(a)$ and defining $\mathcal{F}_1(a) = \frac{F_1(a)}{H(a)}$, $\mathcal{F}_2(a) = \frac{F_2(a)}{H(a)}$,

$$\mathcal{F}_1(a)G_1(x) + \mathcal{F}_2(a)G_2(x) = 1\,. \tag{A.1}$$

Since $x$ and $a$ are real variables, the above expression must be valid for infinitely many values of both $x$ and $a$. Let us consider $a_1$ and $a_2$, two arbitrarily chosen values of $a$. For them, we have

$$\begin{aligned} \mathcal{F}_1(a_1)G_1(x) + \mathcal{F}_2(a_1)G_2(x) &= 1\,, \\ \mathcal{F}_1(a_2)G_1(x) + \mathcal{F}_2(a_2)G_2(x) &= 1\,. \end{aligned} \tag{A.2}$$

Imagine a plane where $G_1(x)$ and $G_2(x)$ are $x$ and $y$ coordinates respectively. Each of the equalities expressed in Eq. (A.2) is represented by a line on this $G_1$–$G_2$ plane. If these two lines intersect, we will have a solution for both $G_1(x)$ and $G_2(x)$: contrary to our hypothesis of irreducibility each would be a determined constant. This means that the matrix

$$\begin{bmatrix} \mathcal{F}_1(a_1) & \mathcal{F}_2(a_1) \\ \mathcal{F}_1(a_2) & \mathcal{F}_2(a_2) \end{bmatrix}$$

should be noninvertible; i.e. its determinant must vanish. This implies that we must have the lines parallel; i.e. both must have the same slope:

$$\frac{\mathcal{F}_2(a_1)}{\mathcal{F}_1(a_1)} = \frac{\mathcal{F}_2(a_2)}{\mathcal{F}_1(a_2)}\,. \tag{A.3}$$

In other words, the ratio $\frac{\mathcal{F}_2(a)}{\mathcal{F}_1(a)}$ is independent of the argument $a$, and hence must be equal to an $a$-independent constant.

This proof can be extended to higher-dimensional spaces for any of the three-, four- or five-term irreducible expressions we have calculated in this paper.

## References


1. F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore, 2001).
2. G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer, 1996).
3. O. L. DeLange and R. E. Raab, *Operator Methods in Quantum Mechanics* (Oxford University Press, Oxford, 1991).
4. A. Lahiri, P. K. Roy and B. Bagchi, *Int. J. Mod. Phys. A* **5**, 1383 (1990).
5. R. A. Leacock and M. J. Padgett, *Phys. Rev. Lett.* **50**, 3 (1983).
6. R. A. Leacock and M. J. Padgett, *Phys. Rev. D* **28**, 2491 (1983).
7. R. A. Leacock and M. J. Padgett, *Phys. Rev. A* **33**, 2775 (1986).
8. R. A. Leacock and M. J. Padgett, *Am. J. Phys.* **55**, 261 (1986).
9. E. Gozzi, *Phys. Lett. B* **165**, 351 (1985).
10. L. Infeld and T. E. Hull, *Rev. Mod. Phys.* **23**, 21 (1951).
11. L. E. Gendenshtein, *JETP Lett.* **38**, 356 (1983).
12. L. E. Gendenshtein and I. V. Krive, *Sov. Phys. Usp.* **28**, 645 (1985).
13. F. Cooper, J. Ginocchio and A. Khare, *Phys. Rev. D* **36**, 2458 (1987).
14. A. Gangopadhyaya, J. V. Mallow and U. P. Sukhatme, *Phys. Rev.* **58**, 4287 (1998).
15. A. Shabat, *Inverse Problems* **8**, 303 (1992).
16. D. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta and U. Sukhatme, *Phys. Rev. A* **48**, 2786 (1993).
17. U. P. Sukhatme, C. Rasinariu and A. Khare, *Phys. Lett. A* **234**, 401 (1997).
18. M. Daoud and M. Kibler, *Phys. Lett. A* **321**, 147 (2004).
19. M. Daoud and M. Kibler, *J. Math. Phys.* **47**, 122108 (2006).
20. R. Dutt, A. Khare and U. P. Sukhatme, *Am. J. Phys.* **56**, 163 (1988).