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Exactly solvable problems of quantum mechanics and their spectrum generating algebras: A review

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Abstract: In this review, we summarize the progress that has been made in connecting supersymmetry and spectrum generating algebras through the property of shape invariance. This monograph is designed to be used by our fellow researchers, by other interested physicists, and by students at the graduate and even undergraduate levels who would like a brief introduction to the field.

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1 Introduction

Using the formalism of supersymmetric quantum mechanics, researchers (including the authors) have shown that all exactly solvable problems of non-relativistic quantum mechanics possess a spectrum generating algebra, also known as a potential algebra.

The main idea is to find a suitable algebra, usually an $so(2,1)$ algebra or some of its nonlinear deformations, and make a connection between the differential realizations of its generators and the physical system. Suppose that we have carefully crafted an algebra

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generated by J_3, J_+ and J_- such that the Hamiltonian of the system is proportional to the product J_+J_-. Then, from knowing the represention of this algebra, we can obtain immediately the spectrum of the Hamiltonian. Another approach is to find differential realizations of the generators of an algebra such that the Hamiltonian of the system is directly proportional to one of its generators, usually J_3. Then, one gets the eigenenergies from knowing the eigenvalues of J_3 in this representation.

In this monograph, we will focus on the first approach, and prove that there is a systematic way of building algebras for shape invariant systems.

The paper is organized as follows. First, we briefly review supersymmetric quantum mechanics, with worked examples. We then summarize the progress that has been made in connecting supersymmetry to spectrum generating algebras, through the requirement of a property of the potential known as shape invariance. We present the algebraic analysis in a generalized form for all solvable systems, for both unbroken and broken supersymmetry. We give examples of their use in the determination of the eigenspectra of typical systems.

2 Supersymmetric quantum mechanics

It is well known that there are two competing methods for determining eigenvalues and eigenfunctions of a simple harmonic oscillator. The first method is to directly solve the Schrödinger equation:

$$-\frac{d^2\psi(x)}{dx^2}+\frac{1}{4}\omega^2x^2\psi(x)=E\psi(x). \tag{1}$$

(We choose $2m=\hbar=1$.) The full machinery for solving this second order differential eigenvalue equation is then employed. The second method, originally given by P.A.M. Dirac [1], consists of writing the Hamiltonian as a product of raising and lowering operators $a^\dagger=-\frac{d}{dx}+\frac{1}{2}\omega x$ and $a=\frac{d}{dx}+\frac{1}{2}\omega x$; whence $H_-=a^\dagger a+\frac{1}{2}\omega$. The ground state is determined by solving the very simple first order differential equation, $a\psi_0(x)=0$, yielding $\psi_0(x)\propto e^{-\int\frac{1}{2}\omega\,x\,dx}=e^{-\frac{1}{4}\omega^2x^2}$.

Higher states are obtained from $a^\dagger\psi_n(x)=\sqrt{n+1}\,\psi_{n+1}(x)$, with energy levels $E_n=\left(n+\frac{1}{2}\right)\omega$. One could equally well have used the Hamiltonian $H_+\equiv a\,a^\dagger-\frac{1}{2}\omega$ since $a\,a^\dagger$ and $a^\dagger a$ differ by unity. Standard quantum mechanics texts make use of both forms to obtain the spectrum of the harmonic oscillator.

Note that if we had simply shifted our definition of H_- down by $\frac{1}{2}\omega$; viz., $H_-=a^\dagger a$, the ground state energy would be 0.

Supersymmetric quantum mechanics (SUSYQM) [2–6] generalizes this second method to all exactly solvable problems of quantum mechanics. As in the harmonic oscillator case, this method involves writing the Hamiltonian as a product of two first order differential operators A^+ and A^-, such that $H=A^+A^-+E_0$ where E_0 is the ground state energy, and

$$A^-(x,a)=\frac{d}{dx}+W(x,a)\quad;\quad A^+(x,a)=-\frac{d}{dx}+W(x,a). \tag{2}$$

The function $W(x,a)$ is a real function of x and a parameter a, and is known as the

superpotential of the problem. This is the generalization of the harmonic oscillator, for which $W(x,a) = \frac{1}{2}\omega\, x$.

Operators A^+ and A^- are Hermitian conjugates of each other: $A^+ = (A^-)^\dagger$. Let us define $H_- \equiv A^+A^- = -\frac{d^2}{dx^2} + V_-$, and $H_+ \equiv A^-A^+ = -\frac{d^2}{dx^2} + V_+$. In general H_- and H_+ are two different Hamiltonians. These Hamiltonians are semi-positive definite, which means that their eigenvalues are greater than or equal to zero. This can be explicitly seen from the expectation value of the Hamiltonian: $\langle\psi|H_-|\psi\rangle = \langle\psi|A^+A^-|\psi\rangle = \|A^-|\psi\rangle\|^2 \geq 0$, and similarly for H_+. In addition, as we shall show below, they have the property of iso-spectrality; i.e., they have the *same* set of energy eigenvalues (except possibly for the ground state) even though they correspond in general to two very different potentials V_- and V_+.

H_- and H_+ are known as partner Hamiltonians in SUSYQM. They are given explicitly by:

$$\begin{aligned} H_\mp = A^\pm\, A^\mp &= \left(\mp\frac{d}{dx} + W(x,a)\right)\left(\pm\frac{d}{dx} + W(x,a)\right) \\ &= \left(-\frac{d^2}{dx^2} + W^2(x,a) \mp \frac{dW(x,a)}{dx}\right) \\ &= \left(-\frac{d^2}{dx^2} + V_\mp(x,a)\right), \end{aligned} \tag{3}$$

where the potentials V_- and V_+ are defined by the superpotential: $V_\mp(x,a) \equiv W^2(x,a) \mp \frac{dW(x,a)}{dx}$.

It is not very hard to find the ground state of the Hamiltonian H_-. An explicit calculation yields:

$$H_-\ \left(\mathrm{e}^{-\int_{x_0}^{x} W(x)dx}\right) = \left(-\frac{d^2}{dx^2} + W^2(x,a) - \frac{dW(x,a)}{dx}\right)\left(\mathrm{e}^{-\int_{x_0}^{x} W(x)dx}\right) = 0.$$

I.e., if $\psi_0^- \sim \mathrm{e}^{-\int_{x_0}^{x} W(x)dx}$ is normalizable, it is an eigenstate of H_-, with eigenvalue zero. This choice of $H_- = A^+A^-$, the same that we suggested earlier for the harmonic oscillator, implies choosing $E_0 = 0$: a shift of the ground state energy of the usual Schrödinger equation value. This convention does not of course affect the physics; it is, however, an essential feature of supersymmetry. Indeed, it will turn out to distinguish between cases of unbroken and broken supersymmetry. The latter does not permit the choice $E_0 = 0$, as we shall discuss below.

Since $H_- = A^+A^-$ is semi-positive definite, this zero energy state is its ground state. Similarly, one can show that $\psi_0^+ \sim \exp\left(\int_{x_0}^{x} W(x)\, dx\right)$ solves the equation. $H_+\psi_0^+ = 0$; i.e., it also has the possibility of being an eigenstate of H_+ with eigenvalue zero. These follow from $\left(\pm\frac{d}{dx} + W(x)\right) e^{\left(\mp\int_{x_0}^{x} W(x)\ dx\right)} = 0$. However, if ψ_0^- is normalizable then ψ_0^+ is not, as they are inverses of each other. If either of them is normalizable, we say that supersymmetry (SUSY) is unbroken. If neither is normalizable, the supersymmetry is said to be broken, and a ground state with a zero eigenvalue does not exist. Thus, the

existence of a zero energy ground state is a necessary and sufficient condition for unbroken supersymmetry. Whether the SUSY is broken or not is therefore completely dictated by the behavior of the superpotential $W(x,a)$. Here, we assume ψ_0^- to be normalizable and supersymmetry to be unbroken. In Sec. 4, we will discuss cases of broken supersymmetry.

To show the iso-spectrality mentioned above for all but the ground state ($n=0$), let us denote by $\psi_n^\pm$ the eigenfunctions of $H_\pm$ that correspond to eigenvalues $E_n^\pm$; $n=1,2,\cdots$.

$$H_+\left(A^-\psi_n^-\right)=A^-A^+\left(A^-\psi_n^-\right)=A^-\left(A^+A^-\psi_n^-\right)=A^-H_-\left(\psi_n^-\right)=E_n^-\left(A^-\psi_n^-\right). \quad (4)$$

Thus $A^-\psi_n^-$ is an eigenstate of H_+, and the corresponding eigenvalue is E_n^-. Hence, except for the H_- ground state which obeys $A^-\psi_0^-=0$, all excited states ψ_n^- of H_- have a one-to-one correspondence with ψ_{n-1}^+: $\psi_{n-1}^+=cA^-\psi_n^-$ and $E_{n-1}^+=E_n^-$, where $n=1,2,\cdots$. The normalization constant c may be obtained as follows:

$$\begin{aligned}1&=\int \psi_n^{+^*}\psi_n^+\,dx=|c|^2\int \psi_{n+1}^{-^*}\left(A^\dagger A^-\right)\psi_{n+1}^-\,dx\\&=|c|^2\int \psi_{n+1}^{-^*}H_+\psi_{n+1}^-\,dx=|c|^2E_{n+1}^-=|c|^2E_n^+.\end{aligned}$$

Thus,

$$\psi_n^+=\frac{1}{\sqrt{E_{n+1}^-}}A^-\psi_{n+1}^-. \quad (5)$$

Similarly, we have

$$\psi_n^-=\frac{1}{\sqrt{E_{n-1}^+}}A^+\psi_{n-1}^+. \quad (6)$$

Thus, if we knew the eigenvalues and eigenfunctions of either of the two partner potentials, we could determine the spectrum of the other. As an example let us demonstrate this method for the unlikely pair of partner potentials, the infinite well and cosec^2x, also known as the trigonometric Pöschl-Teller, or Pöschl-Teller I potential. We begin by showing that they are indeed superpotential partners. Consider a superpotential $W(x)=-b\cot x$ with $b>0$, and $0<x<\pi$. The supersymmetric partner potentials generated by this superpotential are:

$$\begin{aligned}V_-(x,b)&=W^2(x)-\tfrac{dW}{dx}=b(b-1)\,\mathrm{cosec}^2x-b^2\\&\quad\text{and}\\V_+(x,b)&=W^2(x)+\tfrac{dW}{dx}=b(b+1)\,\mathrm{cosec}^2x-b^2.\end{aligned} \quad (7)$$

Now for the special case of $b=1$, the potential $V_-(x,1)$ is a trivial constant function $-b^2=-1$; i.e., just an infinite one-dimensional square well potential whose bottom is set to -1, while its partner potential $V_+(x,1)$ is given by $2\,\mathrm{cosec}^2x-1$. Thus, in general, two supersymmetric partner potentials could be very different. Since we can find out the eigenvalues and eigenfunctions of the infinitely deep square well potential from any book on introductory quantum mechanics, SUSYQM automatically gives us the spectrum

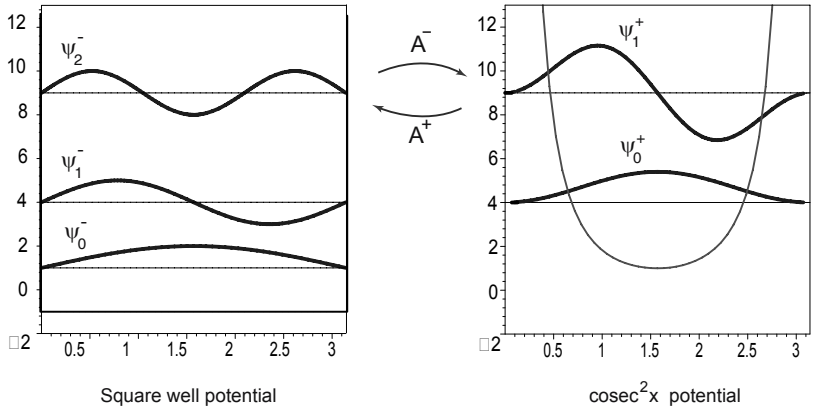

Fig. 1 Square well and its $\mathrm{cosec}^2 x$ partner potential, together with a few energy levels and the corresponding eigenfunctions.

of the very nontrivial $\mathrm{cosec}^2 x$ potential. The eigenspectrum (in simplified units) of the square well potential $V_-(x,1)$ is given by $\psi_n^- \sim \sin(nx)$ and $E_n^- = n^2 (n = 0, 1, 2, \dots)$. Hence, using $\psi_{n-1}^+ \sim A^- \psi_n^-$ and Eq. (2) the eigenspectrum of the $\mathrm{cosec}^2 x$ potential is given by $\psi_{n-1}^+ \sim \left(\frac{d}{dx} - \cot x\right) \sin(nx)$ and $E_n^+ = n^2 (n = 1, 2, 3, \dots)$. We have depicted these two potentials, their low lying eigenstates and corresponding eigenvalues in Fig. (1). For a list of all known solvable potentials, see Ref. [5].

A very interesting application of this isospectrality of the two supersymmetric partner Hamiltonians provides a natural explanation for many cases of "accidental degeneracies" [7]. By defining two operators, Q and $Q^\dagger$, as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \quad \text{and} \quad Q^\dagger = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

we get the following result:

$$\frac{1}{2}\left[Q\,Q^\dagger + Q^\dagger\,Q\right] \equiv \frac{1}{2}\{Q, Q^\dagger\} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}.$$

Thus, in terms of supercharge Q and its adjoint $Q^\dagger$, both Hamiltonians H_- and H_+ can be viewed as components of a supersymmetric Hamiltonian $H = \frac{1}{2}\{Q, Q^\dagger\}$. In Ref. [7], using two explicit examples, Balantekin has shown that the supersymmetric properties of H, that is, the isospectrality of H_- and H_+ can be interpreted as a discrete symmetry in values of a set of parameters.

For example, a three-dimensional harmonic oscillator with spin-orbit coupling

$$H = \frac{1}{2}\left(\vec{p}^{\,2} + \vec{r}^{\,2}\right) + \lambda\left(\vec{\sigma}\cdot\vec{L} + \frac{3}{2}\right) \tag{8}$$

has the following eigenspectrum:

$$E^{\lambda}_{nlj} = 2n + \begin{cases} (1+\lambda)j + (1+\lambda) & \text{for } \; j = l + \frac{1}{2} \\ (1-\lambda)j + 2 & \text{for } \; j = l - \frac{1}{2} \end{cases} . \tag{9}$$

In the above, $n = 0, 1, 2 \cdots$ is the radial quantum number. For a fixed value of j, there is a degeneracy: ($\lambda = 1,\ l = j + \frac{1}{2}$) and ($\lambda = -1,\ l = j - \frac{1}{2}$) both correspond to the same energy $E_n = 2n + 2j + 2$. Two Hamiltonians with these two sets of parameters can be identified with supersymmetric partner H_- and H_+, which are components of a supersymmetric Hamiltonian H. Furthermore, this supersymmetric Hamiltonian is part of one representation of an underlying superalgebra characterized by the parameter j. For further details, see Ref. [7].

3 Shape invariance

From the previous section, we have seen that if the eigenvalues and eigenfunctions of H_- were already known, one would obtain the eigenvalues and the eigenfunctions of H_+, which is in general a completely different Hamiltonian. At this point, we could thus obtain the E^+'s and ψ^+'s from the E^-'s and ψ^-'s, or vice versa, but we can go no further. However, if the superpotential of the problem obeys a further constraint, "shape invariance" [8, 9], then for either Hamiltonian we can derive all the eigenvalues and, step-by-step, construct all the eigenfunctions; i.e., the problem becomes exactly solvable, as we shall now show.

A superpotential is said to be shape invariant if it satisfies the condition

$$W^2(x, a_0) + \frac{dW(x, a_0)}{dx} + g(a_0) = W^2(x, a_1) - \frac{dW(x, a_1)}{dx} + g(a_1). \tag{10}$$

In other words, for a parameter a_0, there is another parameter a_1 such that $V_+(x, a_0)$ can be written as $V_-(x, a_1)$ plus an additive constant $g(a_1) - g(a_0) \equiv R(a_0)$ [‡]. Thus, shape invariant partner potentials $V_+(x, a_0)$ and $V_-(x, a_1)$ have the same x-dependence. In terms of Hamiltonians, the shape invariance condition reads

$$H_+(x, a_0) + g(a_0) = H_-(x, a_1) + g(a_1). \tag{11}$$

Now, let us see how shape invariance leads to exact solvability. Suppose we want to find the spectrum of $H_-(x, a_0)$. We know that $E^-_0(a_1) = 0$. Let us find, e.g., $E^-_2(a_0)$. As we have shown in the previous section, it is equal to $E^+_1(a_0)$. But from shape invariance, $E^+_1(a_0) = E^-_1(a_1) + g(a_1) - g(a_0)$. Going one more step, one has $E^-_1(a_1) = E^+_0(a_1)$, which in turn from shape invariance is equal to $E^-_0(a_2) + g(a_2) - g(a_1)$, where a_2 is another parameter. However, if the Hamiltonian $H_-(x, a_2)$ corresponds to unbroken SUSY we are guaranteed that the ground state energy of the system $E^-_0(a_2) = 0$ and the corresponding eigenfunction is $\psi^-_0(x, a_2) = N \exp\left(-\int_{x_0}^{x} W(x, a_2)dx\right)$, where N is

[‡] This is only true for $a_1 \neq a_0$. For the harmonic oscillator, $a_1 = a_0$, and $R(a_0)$ has an explicit nonzero value.

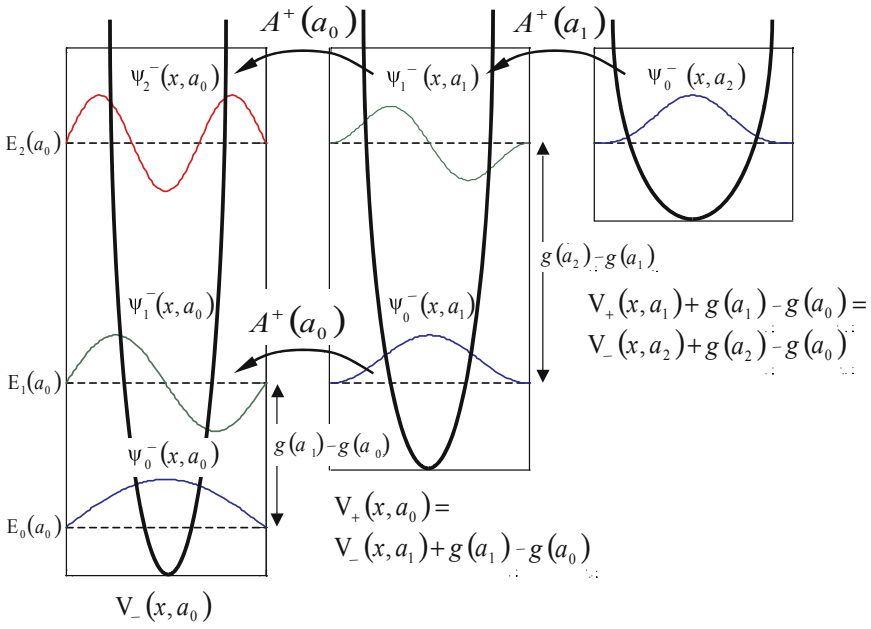

Fig. 2 Shape invariant potentials, their energy levels and the corresponding eigenfunctions.

determined by normalization once W is given. Hence, we have

$$E_2^-(a_0) = g(a_2) - g(a_1) + g(a_1) - g(a_0) = g(a_2) - g(a_0).$$

Using Eq. (6), the corresponding eigenfunction is obtained as follows:

$$\begin{aligned}
\psi_2^-(x,a_0) &= \frac{1}{\sqrt{E_1^+(a_0)}}\, A^+(x,a_0)\,\psi_1^+(x,a_0) \\
&= \frac{1}{\sqrt{E_1^+(a_0)}}\, A^+(x,a_0)\,\psi_1^-(x,a_1) \\
&= \frac{1}{\sqrt{E_1^+(a_0)\,E_0^+(a_1)}}\, A^+(x,a_0)\,A^+(x,a_1)\,\psi_0^+(x,a_1) \\
&= \frac{1}{\sqrt{E_1^+(a_0)\,E_0^+(a_1)}}\, A^+(x,a_0)\,A^+(x,a_1)\,\psi_0^-(x,a_2). \qquad (12)
\end{aligned}$$

All eigenfunctions and eigenvalues of the Hamiltonian $H_-(x,a_0)$ can be determined by an iteration of this algorithm. Thus, SUSYQM and shape invariance provide an excellent formalism to determine the entire spectrum of solvable quantum systems through a step-by-step algebraic procedure, without any need to solve a differential equation.

In the last section, we happened to know the spectrum of the infinite square well from ordinary quantum mechanics and we used that to determine the spectrum of its partner, the $\mathrm{cosec}^2 x$ potential. Now we demonstrate that they are in fact shape invariant partners and thus, we can determine the spectrum of either without a priori knowledge of the other. The potential $V_+(x,b)$ from Eq. (7) is

$$V_+(x,b) = W^2(x) + \frac{dW}{dx} \;=\; b(b+1)\,\mathrm{cosec}^2 x - b^2,$$

which can be rewritten as

$$\begin{aligned} V_+(x,b) &= (b+1)[(b+1)-1]\,\mathrm{cosec}^2 x - (b+1)^2 + (b+1)^2 - b^2 \\ &= V_-(x,b+1) + (b+1)^2 - b^2. \end{aligned} \tag{13}$$

So the potential $V_-(x,b)$ is a shape invariant potential as defined in Eq. (10), with $a_0 = b$ and $a_1 = a_0 + 1 = b + 1$ and therefore $g(b) = b^2$. Here we solve the infinite square well as an example. As we saw earlier, setting $b = 1$ leads to $V_+(x,1) = 2\mathrm{cosec}^2 x - 1$ and $V_-(x,1) = -1$ with the latter representing an infinitely deep potential well in the region $0 < x < \pi$. The ground state eigenfunction of $H_-(x,1)$ is $\psi_0^-(x,1) \sim e^{-\int W(x,1)dx} \sim e^{\int \cot x dx} \sim \sin x$; its energy is, as usual, 0. Now, we use shape invariance to determine the excited states.

From Eq. (13), $V_+(x,1) = V_-(x,2) + 3$; thus, $g(2) - g(1) = 3$. Using the fact that the ground state energy $E_0^-(2)$ of $H_-(x,2)$ is zero (like all H_- ground states where ψ_0^- is normalizable), we find that the ground state energy of $H_+(x,1)$ is $E_0^+(1) = 3$. But this is just the energy of the first excited state of H_-, our 'target' Hamiltonian.

The common ground state eigenfunction of $H_+(x,1)$ and $H_-(x,2)$ is given by $\psi_0^+(x,1) = \psi_0^-(x,2) \sim e^{-\int W(x,2)dx} \sim e^{\int 2\cot x dx} \sim \sin^2 x$. Then the first excited state of $H_-(x,1)$ is given by $\psi_0^-(x,1) \sim A^\dagger(x,1)\sin x = \left(-\frac{d}{dx} - \cot x\right)\sin^2 x \sim \sin 2x$. Thus, we have derived the energy and the eigenfunction of the first excited state of $H_-(x,1)$. By iterating this procedure, we can generate the entire spectrum. In particular, the eigenvalues are given by $E_n^-(b) = (b+n)^2 - b^2$.

In the above example, our parameters from Eqs. (10) and (11) were $a_0 = b$ and $a_1 = b + 1 = a_0 + 1$. This type of shape invariance, for which $a_n = a_{n-1} +$ constant, is known as additive shape invariance. In contrast, shape invariance conditions in which parameters are related by $a_1 = q\,a_0$ are called multiplicative shape invariance. Potentials with multiplicative shape invariance are only known as series, not in closed form [10], and we will discuss them later.

At this point, we would like to emphasize that shape invariance does not always help one in determining the spectrum. There is another important ingredient necessary, and that is unbroken supersymmetry [11]. We will assume that supersymmetry is unbroken in all that follows. In Section 7, we will discuss cases of broken supersymmetry.

All known exactly solvable closed form potentials have been demonstrated by numerous authors [12–16], starting with Pauli and Fock [17], to possess a spectrum generating algebra (SGA). We now show that there is a connection between SGA and shape invariance of these systems.

4 Connection to algebra

One of the nontrivial algebras with which most physics students are familiar is the $so(3)$ algebra generated by the Cartesian components of the angular momentum operators L_1, L_2 and L_3. Their commutator is given by

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k. \tag{14}$$

We usually work in a basis where L^2 and L_3 are diagonal. In this basis, common eigenstates of L^2 and L_3 are connected by raising and lowering operators $L_\pm = L_1 \pm i\, L_2$, which obey

$$[L_3, L_\pm] = \pm L_\pm \tag{15}$$
$$[L_+, L_-] = 2L_3. \tag{16}$$

The eigenvalues of L^2 and L_3 are given by $l(l+1)$ and m respectively (with $\hbar = 1$.) In the coordinate representation, $L_\pm$ and L_3 are represented by the following differential operators:

$$L_\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right), \quad L_3 = -i\left(\frac{\partial}{\partial\phi}\right) \tag{17}$$

Now we would like to show that there are quantum mechanical systems that correspond to this algebra. As one example let us make a change of variable

$$z = \ln\left[\tan\left(\frac{\theta}{2}\right)\right]. \tag{18}$$

This change of variable maps the $(0, \pi)$ domain to $(-\infty, \infty)$. In these new coordinates L_+ and L_- become

$$L_\pm = e^{\pm i\phi}\left(-i\sinh z\,\frac{\partial}{\partial\phi} \pm \cosh z\,\frac{\partial}{\partial z}\right), \tag{19}$$

while L_3 remains unchanged. Like the A^+A^- operator we have seen before, we would like to see if we can associate the operator L_+L_- with a quantum mechanical Hamiltonian. Its eigenvalue equation $L_+L_-\psi = \lambda\psi$ yields,

$$\left[-\sinh^2 z\,\frac{\partial^2}{\partial\phi^2} - i\frac{\partial}{\partial\phi} - \cosh^2 z\,\frac{\partial^2}{\partial z^2}\right]\psi(z,\phi) = \lambda\,\psi(z,\phi). \tag{20}$$

Dividing this equation by $\cosh^2 z$ and rearranging the terms, we get

$$\left[-\frac{\partial^2}{\partial z^2} + \left(-i\,\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} - \lambda\right)\operatorname{sech}^2 z\right]\psi(z,\phi) = \frac{\partial^2}{\partial\phi^2}\psi(z,\phi). \tag{21}$$

From Eq. (17), the eigenfunctions of the operator L_3 are given by $f(z)\,e^{i\,m\,\phi}$. Acting on these eigenfunctions, Eq. (21) yields

$$\left[-\frac{\partial^2}{\partial z^2} + \left(m - m^2 - \lambda\right)\operatorname{sech}^2 z\right] f(z) = -m^2 f(z). \tag{22}$$

Note that this is now a one-dimensional Schrödinger equation. The m's are "encoding" the behavior of $\frac{\partial}{\partial\phi}$'s on the eigenfunctions of L_3, $f(z)\,e^{i\,m\,\phi}$.

Since $L_+L_- = L^2 - L_z^2 + L_z$, the eigenvalue λ from Eq. (20) is given by $\lambda = l(l+1) - m^2 + m$. Substituting this expression in Eq. (22) leads to the following Schrödinger equation:

$$\left[-\frac{\partial^2}{\partial z^2} - l(l+1)\operatorname{sech}^2 z\right] f(z) = -m^2 f(z), \tag{23}$$

where $V(z,m,l) = -l(l+1)\,\mathrm{sech}^2 z$ is the potential and $-m^2$ is the corresponding eigenenergy. To connect this problem to SUSYQM, we will show that there is a superpotential for which $V(z,m,l) = V_-(z,m,l) +$ a constant. Since a "$\mathrm{sech}^2 z$" term appears in the potential, we try the ansatz $W = a\,\tanh\,z$. This leads to $W^2 - W' = (-a^2 - a)\,\mathrm{sech}\,z^2 + a^2$. This superpotential generates the potential $V_-(z) = -l(l+1)\,\mathrm{sech}^2 z + l^2$, provided $a = l$. Thus, $V_-(z) = V(z,m,l) + l^2$, and Eq. (23), with $l^2 f(z)$ added to both sides, becomes

$$\left[-\frac{\partial^2}{\partial z^2} - l(l+1)\,\mathrm{sech}^2 z + l^2\right] f(z) = (l^2 - m^2) f(z). \tag{24}$$

Now we have a quantum mechanical system described by the Hamiltonian $H = -\frac{\partial^2}{\partial z^2} - l(l+1)\,\mathrm{sech}^2 z + l^2$, familiar to physicists as the Pöschl-Teller II potential and studied in, among many other places, Ref. [18], where it was shown to be generated by the

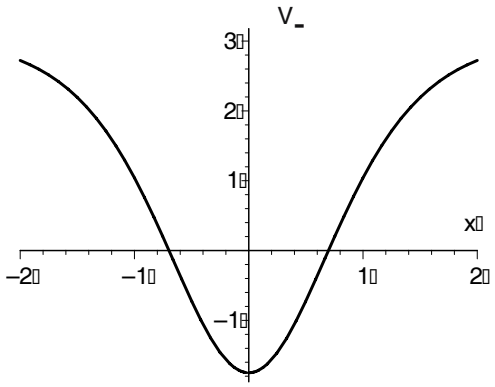

Fig. 3 Pöschl-Teller potential for $l = 1.75$.

superpotential $W = l\,\tanh\,z$. (N.B., this is not the same as the trigonometric Pöschl-Teller I potential we studied in the introduction.) Thus we have an example where we started from a very well known algebra, and showed that it is connected to a solvable one-dimensional Schrödinger equation through the transformation in Eq. (18). Other transformations will produce other potentials.

In the next section, we will show that all shape invariant problems have an underlying algebra. We will explicitly construct these algebras.

5 The algebraic shape invariant model

In Sections 2 and 3 we learned about SUSYQM and how the additional property of shape invariance, if present in the model, renders it completely solvable algebraically. In Section 4, we introduced the well known $so(3)$ algebra and showed how it leads to a shape invariant potential. In particular, we showed how parameters of the Pöschl-Teller I potential are generated from eigenvalues of the L_3 operator. At this point it is natural to wonder if there exists a deeper principle that connects the shape invariance condition to the solvability of a model. This question has been resolved by the work of several groups [19–23].

5.1 Building the algebra of shape invariance

We will now establish that there is always an algebra behind every shape invariant problem. The process is as follows.

Knowing that the shape invariant Hamiltonians $H_- = A^+A^-$ and $H_+ = A^-A^+$ are connected via

$$H_+(z,a_0) + g(a_0) = H_-(z,a_1) + g(a_1); \quad a_1 = \eta(a_0), \tag{25}$$

we rewrite this shape invariance condition in terms of A^+ and A^- such that it resembles a commutation relation

$$A^+(z,a_1)A^-(z,a_1) - A^-(z,a_0)A^+(z,a_0) = f(a_0) \tag{26}$$

where $f(a_0) = g(a_0) - g(a_1) \equiv g(a_0) - g(\eta(a_0))$. Then we build the algebra corresponding to the shape invariant system (26) by defining a set of three differential operators J_+, J_- and J_3 that satisfy the following conditions:

(1) The algebra generated by the set $\{J_+, J_-, J_3\}$ is closed. This means that all the commutators among the generators should be expressed only in terms of the generators themselves. Using the analogy with the algebra of the angular momentum let us define the following commutation relations:

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = F(J_3). \tag{27}$$

(Note that when $F(J_3) = 2J_3$ we obtain the well-known algebra of angular momentum discussed in Section 4.)

(2) The closure of the algebra must be related to the shape invariance of the system, Eq. (26). Therefore, we require that acting with the third commutation relation of Eq. (27) on a set of eigenfunctions of J_3, recovers Eq. (26).

From the conditions above, we see that we have somehow to encode the information about A^+ and A^- into the definition of J_+ and J_- respectively. These observations plus our experience with traditional coordinate representations of $L_\pm$ and L_3 (Eq. (17)) suggest the following ansatz for J_+ and J_-:

$$J_+ = \mathrm{e}^{is\phi}\mathcal{A}^+, \quad J_- \equiv (J_+)^\dagger = \mathcal{A}^-\,\mathrm{e}^{-is\phi}, \tag{28}$$

where s is a constant parameter. The operator $\mathcal{A}^-$ is obtained from $A^- \equiv A^-(z,a_0)$ by introducing an auxiliary variable ϕ independent of z and replacing the parameter a_0 by an operator function $\chi(i\partial_\phi)$:

$$A^- \equiv A^-(z,a_0) \xrightarrow{a_0 \mapsto \chi(i\partial_\phi)} \mathcal{A}^- \equiv \mathcal{A}^-(z,\chi(i\partial_\phi)). \tag{29}$$

Similarly, $\mathcal{A}^+ = (\mathcal{A}^-)^\dagger$. Thus, the operators $\mathcal{A}^\pm$ encode the relation between $\partial_\phi \equiv \frac{\partial}{\partial\phi}$ and its eigenvalue a_0, and $J_\pm$ "behave" like $L_\pm$ of Eq. (19).

The reason for the substitution (29) will become evident after this short calculation:

$$\begin{aligned}[J_+, J_-] &= J_+J_- - J_-J_+ \\ &= \mathrm{e}^{is\phi}\mathcal{A}^+(z,\chi(i\partial_\phi))\,\mathcal{A}^-(z,\chi(i\partial_\phi))\,\mathrm{e}^{-is\phi} - \mathcal{A}^-(z,\chi(i\partial_\phi))\,\mathcal{A}^+(z,\chi(i\partial_\phi)) \\ &= \mathcal{A}^+(z,\chi(i\partial_\phi+s))\,\mathcal{A}^-(z,\chi(i\partial_\phi+s)) - \mathcal{A}^-(z,\chi(i\partial_\phi))\,\mathcal{A}^+(z,\chi(i\partial_\phi)). \quad (30)\end{aligned}$$

In the last line, we have used the fact that the operator $i\partial_\phi\,\phi$ acting on a arbitrary function $f(\phi)$ yields $i\partial_\phi\,\phi f(\phi) = \phi\ (i\partial_\phi + i)\,f(\phi)$. This implies $(i\partial_\phi)\,\mathrm{e}^{-is\phi} = \mathrm{e}^{-is\phi}\,(i\partial_\phi + s)$. Observe now that the right hand side of Eq.(30) is similar to the left hand side of Eq. (26) provided that we make the following mappings:

$$a_0 \mapsto \chi(i\partial_\phi), \qquad a_1 \mapsto \chi(i\partial_\phi + s). \quad (31)$$

But we know that $a_1 = \eta(a_0)$; therefore this identification is valid as long as we judiciously choose the function $\chi(i\partial_\phi)$ such that

$$\chi(i\partial_\phi + s) = \eta\left(\chi(i\partial_\phi)\right). \quad (32)$$

Eq. (32) is a rather general, indeed generous constraint. Let us look at some examples to illustrate it. In the case of Eq. (13), the change of parameters is a simple translation: $a_1 \equiv \eta(a_0) = a_0 + 1$ where $a_0 = b$. We can immediately see that the function that models this translation is the identity function $\chi(z) = z$. Indeed we have in general for translational shape invariance $\chi(z+s) = z + s = \chi(z) + s$, which models perfectly the desired change of parameters for the infinite well and cosec^2 partner potentials if $s = 1$. Similarly, for shape invariant potentials characterized by a scaling change of parameters $a_1 \equiv \eta(a_0) = qa_0$, the function $\chi(z)$ that satisfies Eq. (32) is simply $\chi(z) = \mathrm{e}^z$. We can immediately check that $\chi(z+s) = \mathrm{e}^{z+s} = \mathrm{e}^s\,\mathrm{e}^z = q\,\chi(z)$, where we denoted $\mathrm{e}^s = q$.

Next, we define the operator J_3 such that $[J_3, J_\pm] = \pm J_\pm$. Since $\left[-\frac{i}{s}\,\partial_\phi\,,\,\mathrm{e}^{\pm is\phi}\right] = \pm\,\mathrm{e}^{\pm is\phi}$, by direct calculation we can check that

$$J_3 = k - \frac{i}{s}\,\partial_\phi \quad (33)$$

satisfies these requirements. Here k is an arbitrary constant.

The last step in constructing the algebra for shape invariant systems is to identify the function $F(J_3)$ from (27). Comparing Eq. (26) with its extended form Eq. (30), and writing Eq. (33) as $s\,k - s\,J_3 = i\partial_\phi$, we have

$$f(a_0) \mapsto f\left(\chi(i\partial_\phi)\right) = f\left(\chi(s\,k - sJ_3)\right) \equiv F(J_3). \quad (34)$$

Putting together all the above results, we observe that in fact we have proved the following:

Theorem 5.1. *To any shape invariant system characterized by*

$$A^+(z,a_1)A^-(z,a_1) - A^-(z,a_0)A^+(z,a_0) = f(a_0), \quad a_1 = \eta(a_0) \quad (35)$$

we can associate [24] an algebra generated by

$$J_+ = \mathrm{e}^{is\phi}\,\mathcal{A}^+\left(z,\chi(i\partial_\phi)\right) \tag{36}$$

$$J_- = \mathcal{A}^-\left(z,\chi(i\partial_\phi)\right)\,\mathrm{e}^{-is\phi} \tag{37}$$

$$J_3 = k - \frac{i}{s}\,\partial_\phi \tag{38}$$

satisfying the commutation relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = F(J_3), \tag{39}$$

where

$$F(J_3) = f\left(\chi(s\,k - sJ_3)\right). \tag{40}$$

The function f in Eq. (40) is given by the shape invariance condition (35), while the function χ must satisfy the compatibility equation: $\chi(i\partial_\phi + s) = \eta\left(\chi(i\partial_\phi)\right)$, where η models the change of parameter $a_1 = \eta(a_0)$ of Eq. (35).

To exemplify the above theorem, let us apply it to the Pöschl-Teller II potential studied in the previous section. We expect to obtain the algebra of angular momentum. First, we briefly describe the shape invariance of the Pöschl-Teller II potential. We recall that the potential $V_-(z,l) = -l(l+1)\,\mathrm{sech}^2 z + l^2$ is generated by the superpotential $W(z,l) = l\tanh x$ in the usual way: $V_- = W^2 - W'$. Its supersymmetric partner $V_+ = W^2 + W'$ is $V_+(z,l) = -l(l-1)\,\mathrm{sech}^2 z + l^2$. The shape invariance is now evident if we observe that $V_+(z,l) = V_-(z,l-1) + (2l-1)$. Therefore, in terms of A^+ and A^- operators, the shape invariance for the Pöschl-Teller II potential reads

$$A^+(z,l-1)\,A^-(z,l-1) - A^-(z,l)\,A^+(z,l) = 1 - 2l. \tag{41}$$

Now we can identify the main objects of our model, and build the corresponding shape invariance algebra. We have:

(1) The parameters of the model are $a_0 = l$ and $a_1 = l - 1$.

(2) The change of parameter $a_1 = \eta(a_0)$ is thus given by $a_1 = a_0 - 1$. Therefore we have a translational change of parameter: $a_1 = a_0 + s$ with the translation parameter $s = -1$. Translation implies that the function satisfying (32) is the identity function $\chi(z) = z$.

(3) From the concrete shape invariance condition (41) of the Pöschl-Teller II potential we get $f(a_0) = 1 - 2\,a_0$. Then, the function $F(J_3)$ is given by $F(J_3) = f(\chi(s\,k - sJ_3))) = f(J_3 - k) = 1 - 2(J_3 - k) = -2J_3$ if we choose the arbitrary constant $k = -1/2$.

(4) Defining $J_\pm$ and J_3 as prescribed by Eqs. (36 - 38), we obtain a set of three differential operators satisfying the commutation relations $[J_3, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = -2J_3$. This is the $so(2,1)$ algebra [25]. We can formally recover the $so(3)$ algebra of angular momentum if we define $L_+ = iJ_+, L_- = iJ_-$ and $L_3 = J_3$.

Note that the coordinate realization for the angular momentum generators given by our model is

$$L_+ = i\,\mathrm{e}^{-i\phi}\left(-\frac{d}{dz} + i\partial_\phi \tanh z\right), \quad L_- = i\left(\frac{d}{dz} + i\partial_\phi \tanh z\right)\mathrm{e}^{i\phi}, \quad L_3 = -\frac{1}{2} + i\,\partial_\phi. \tag{42}$$

This realization is different from the "traditional realization" presented in Section 4. Nevertheless, the differential operators we obtained satisfy the commutation relations of the angular momentum algebra, as the reader can readily check. [Note that these are *not* the usual cylindrical coordinate $z's$. Recall that here $z = \ln\left(\tan\frac{\theta}{2}\right)$.]

6 Obtaining the energy spectrum from algebra representations

At this stage we are able to build an algebra of differential operators corresponding to any shape invariant system. We will now exhibit how we can use this algebra to obtain the energy spectrum for the Hamiltonian. From Eq. (30), we observe that

$$J_+ J_- = \mathcal{A}^+\left(z, \chi(i\partial_\phi + s)\right) \mathcal{A}^-\left(z, \chi(i\partial_\phi + s)\right). \tag{43}$$

Acting upon the eigenstates of J_3, operators $i\partial_\phi$ are replaced by their eigenvalues m and the expression above turns into a Hamiltonian $H_-(z, a_1)$. We have

$$J_+ J_-\, e^{-im\phi}\psi(z) = \mathcal{A}^+\left(z, \chi(i\partial_\phi + s)\right) \mathcal{A}^-\left(z, \chi(i\partial_\phi + s)\right)\, e^{-im\phi}\psi(z). \tag{44}$$

From Eq. (29), $\mathcal{A}^+\left(z, \chi(i\partial_\phi + s)\right) \mathcal{A}^-\left(z, \chi(i\partial_\phi + s)\right) \mapsto A^+(z, a_1)\, A^-(z, a_1)$, where a_1 is the parameter associated with $\chi(i\partial_\phi + s)$, as in Eqs. (31). But recall from Eq. (3) that $A^+ A^- = H_-$. Thus

$$J_+ J_-\, e^{-im\phi}\psi(z) = H_-(z, m)\, e^{-im\phi}\psi(z) = E\, e^{-im\phi}\psi(z). \tag{45}$$

The eigenvalues m of J_3 become parameters of the Hamiltonian, and characterize the shape of the potential. Therefore, the spectrum of the operator J_+J_- gives the spectrum of the Hamiltonian. Consequently we need to know the action of individual operators J_+ and J_- respectively on a set of eigenvectors of the operator J_3. In the case of the angular momentum algebra generated by L_+, L_- and L_3, we were looking for a set of commuting operators and their common eigenvectors. The well known choice was L^2 and L_3; their common eigenvectors were denoted by $|\, l, m\rangle$. In this basis L_+ and L_- were raising and lowering operators for the eigenvalue m, and did not affect the eigenvalue l because L^2 commuted with $L_\pm = L_1 \pm iL_2$. In this general case, J_+ and J_- do not necessarily commute with J^2, so we search for an operator that commutes with all generators of the algebra; viz., J_+, J_- and J_3. This operator, known in general as the Casimir of the algebra and denoted by C, would be analogous to the L^2 operator of the angular momentum algebra. For the general algebra described by Eq. (39), it can be explicitly checked that the operator

$$C = J_- J_+ + G(J_3) = J_+ J_- + G(J_3 - 1) \tag{46}$$

with the function G such that

$$F(J_3) = G(J_3) - G(J_3 - 1) \tag{47}$$

does indeed commute with J_+, J_- and J_3, and thus C defined above is the Casimir for our algebra [26]. (For our familiar angular momentum algebra, where $F(J_3) = 2J_3$, the function $G(J_3)$ is given by $J_3^2 + J_3$, and the Casimir is $J_-J_+ + J_3^2 + J_3 = J^2$, as expected.)

In analogy with the angular momentum algebra, we work in a basis in which J_3 and C are diagonal and $J_\pm$ play the role of raising and lowering operators, respectively. Operating on an arbitrary eigenstate $|\, h\rangle$, we have

$$\begin{aligned} J_3|\, h\rangle &= h|\, h\rangle, \\ J_-|\, h\rangle &= a(h)\;|\, h-1\rangle, \\ J_+|\, h\rangle &= a^\star(h+1)\;|\, h+1\rangle, \end{aligned} \tag{48}$$

where we have used $J_+ = (J_-)^\dagger$. To completely identify the action of $J_\pm$ on eigenstates $|\, h\rangle$ of J_3 we need to know these coefficients $a(h)$. Applying the third commutation relation of Eq. (27) to $|\, h\rangle$, we obtain, using Eq. (47),

$$|\, a(h)|^2 - |\, a(h+1)|^2 = G(h) - G(h-1). \tag{49}$$

Now, we will try to determine the representation for this algebra; i.e., determine the allowed values of h and corresponding values for $a(h)$. This is analogous to the determination of eigenvalues m of L_3 for a given value of l for the angular momentum algebra. The number of distinct values of m for a given l is called the dimension of the representation. In the general case, as we shall see shortly, the relationship of $G(h)$ to h determines the dimension of the representation.

Let us say $h = h_{min}$ is the lowest state in a representation. This implies that $J_-|\, h_{min}\rangle = a\,(h_{min})\,|\, h-1\rangle = 0$, which means $a(h_{min}) = 0$. From Eq. (49) we get

$$a(h_{min}+1)^2 = G(h_{min}-1) - G(h_{min}). \tag{50}$$

Similarly, we can determine the coefficient $a(h_{min}+2)^2$. It is just $a(h_{min}+1)^2 + G(h_{min}) - G(h_{min}+1) = G(h_{min}-1) - G(h_{min}+1)$. Iterating this procedure, we can generate a general formula for $a(h)$:

$$a(h_{min}+n)^2 = G(h_{min}-1) - G(h_{min}+(n-1)) \qquad n = 1, 2, \cdots. \tag{51}$$

Substituting $h_{min} + n = h$ leads to

$$a(h)^2 = G(h-n-1) - G(h-1)) \qquad n = 1, 2, \cdots. \tag{52}$$

Operators J_+ and J_- are adjoints of each other, and hence, the operator J_+J_- is positive semi-definite; i.e., its eigenvalues $a(h_{min}+n)^2$ must be positive or zero. This observation has important implications for the representation of the algebra. We started from a state identified by the value of h; viz., h_{min}. Repeated action of J_+ on this state generates states with higher values of h, increasing it by unity every time. There are two possibilities at this point:

a) There is an integer n_{max} such that $J_+ |h_{min} + n_{max}\rangle = a(h_{min} + n_{max} + 1) |h_{min} + n_{max} + 1\rangle = 0$. This would generate a finite dimensional representation of dimension n_{max}, like angular momentum.

b) The raising process never ends and we get an infinite dimensional representation, like the harmonic oscillator.

This gives

$$\begin{aligned} a(h_{min} + n)^2 &= G(h_{min} - 1) - G(h_{min} + (n-1)) \\ &= (h_{min} - 1)^2 + (h_{min} - 1) - (h_{min} + (n-1))^2 - (h_{min} + (n-1)) \\ &= -2n\, h_{min} - n^2 + n, \end{aligned} \tag{53}$$

where we used (as noted following Eq. (47))

$$G(J_3)|h\rangle = J_3(J_3 + 1)|h\rangle = h(h+1)|h\rangle. \tag{54}$$

We first consider Case a). Setting $a(h_{min} + n_{max}) = 0$, we get $h_{min} = (1 - n_{max})/2$. For $n_{max} = 1, 2, \cdots,$

n_{max}	h_{min}	Values of h	Dimension of the representation
1	0	$h = 0$	1
2	$-\frac{1}{2}$	$h = -\frac{1}{2}, \frac{1}{2}$	2
3	-1	$h = 1, 0, -1$	3
4	$-\frac{3}{2}$	$h = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$	4

This is exactly what we expected from our knowledge of angular momentum algebra. We also see that n_{max} gives the dimension of the representation.

Now, using the trigonometric Pöschl-Teller I potential that we described in Section 2 as an example, we will show how to determine the spectrum, this time using the underlying potential algebra. The superpotential for this system is $W(z, b) = -b \cot z$, and, as we have seen before, we have $a_0 = b$ and $a_1 = b + 1 = a_0 + 1 = \eta(a_0)$. Using the set of prescriptions analogous to those following Eq. (41), we take $s = 1$. Next, we replace the parameter b by the operator $i\,\partial_\phi$ to get $W(z, i\,\partial_\phi) = -i\,\partial_\phi \cot z$. Note that the operator $i\,\partial_\phi$ commutes with any function of z and ∂_z since we defined the auxiliary variable ϕ to be independent of z (see Eq. (29)). Using this superpotential operator $W(z, i\,\partial_\phi) = -i\,\partial_\phi \cot z$, we find that the operators

$$J_+ = \mathrm{e}^{i\phi}\left(-\frac{d}{dz} - i\partial_\phi \cot z\right),\ J_- = \left(\frac{d}{dz} - i\partial_\phi \cot z\right)\mathrm{e}^{-i\phi},\ J_3 = k - i\,\partial_\phi$$

for $k = -\frac{1}{2}$ satisfy the algebra

$$[J_3, J_\pm] = \pm J_\pm; \text{ and } \quad [J_+, J_-] = 2J_3.$$

Thus, the potential algebra of this quantum system is $so(3)$, which is the same as the one

generated by angular momentum operators. The operator J_+J_- then is given by

$$\begin{aligned} J_+J_- &= \left(-\frac{d}{dz} - (i\partial_\phi + 1)\,\cot z\right)\left(\frac{d}{dz} - (i\partial_\phi + 1)\,\cot z\right) \\ &= \left\{-\frac{d^2}{dz^2} + (i\partial_\phi)(i\partial_\phi + 1)\,\mathrm{cosec}^2 z - (i\partial_\phi + 1)^2\right\}. \end{aligned}$$

This operator, acting on a state $|h\rangle \propto e^{-ih\phi}\psi(z,h)$ and using $i\partial_\phi|h\rangle = (-\frac{1}{2} - J_3)|b\rangle = -\frac{1}{2} - h)|b\rangle$, leads to the following Schrödinger equation:

$$\begin{aligned} H_-\left(z, -\frac{1}{2} - h + 1\right) &= \left\{-\frac{d^2}{dz^2} + \left(-\frac{1}{2} - h\right)\left(-\frac{1}{2} - h + 1\right)\mathrm{cosec}^2 z - \left(-\frac{1}{2} + 1\right)^2\right\}\psi_n(z,h) \\ &= [G(h-n-1) - G(h-1)]\,\psi_n(z,h) \\ &= [(h-n-1)(h-n) - (h-1)h]\,\psi_n(z,h) \\ &= \left[n^2 - n(2h-1)\right]\psi_n(z,h). \end{aligned}$$

Now, setting $-\frac{1}{2} - h + 1 = b$, we get

$$H_-\,(z,b)\,\psi_n(z,b) = \left[(b+n)^2 - b^2\right]\psi_n(z,b), \tag{55}$$

which are exactly the same eigenvalues as those obtained in Section 2.

So far we have discussed examples of shape invariant potentials with a translational change of parameter: $a_1 = a_0 + s$. Another interesting case is when the change of parameter is given by a scaling relation: $a_1 = q\,a_0$, where q is a constant $(0 < q < 1)$. This choice generates self-similar potentials [10, 22]. We will conclude this section by finding the underlying algebra and the energy spectrum for the potentials generated by a scaling shape invariance.

The scaling transformation can be rewritten as $a_1 = e^s\,a_0$, where $s < 0$. Then, the function χ of Eq. (32) is $\chi(z) = e^z$. The function f of Eq. (35) is simply $f(a_0) = -R_1\,a_0$. Consequently, we get

$$F(J_3) = f(\chi(sk - sJ_3)) = -R_1\,e^{-sJ_3} = -R_1\,q^{-J_3},$$

where we chose the arbitrary constant k of Eq. (40) to be $k = 0$. We have found that the algebra of scaling shape invariance is generated by J_+, J_- and J_3 satisfying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -R_1\,q^{-J_3}. \tag{56}$$

To find the energy spectrum, we should first identify the function $G(J_3)$ of Eq. (47). Then we use this function to find the coefficient $a(h)^2$ of Eq. (52), and hence the energy spectrum. In our case, one gets $G(J_3) = -\frac{R_1}{1-q}\,q^{-J_3}$; hence

$$\begin{aligned} a(h)^2 &= G(h-n-1) - G(h-1) \\ &= R_1\,e^{1-h}\,\frac{q^n - 1}{q-1} \\ &= \alpha(h)\,\frac{q^n - 1}{q-1}, \end{aligned} \tag{57}$$

where $\alpha(h) = R_1\, e^{1-h}$. The spectrum of $H_-(x, a_0)$ is given by $H_-|h\rangle = a(h)^2|h\rangle$; therefore, from Eq. (57) the eigenenergy for a given value of h is

$$E_n(h) = \alpha(h)\,\frac{q^n - 1}{q - 1}; \quad \alpha(h) = R_1\, e^{1-h}, \tag{58}$$

which coincides with the result of Ref. [10].

The spectrum generating algebra (SGA) associated with shape invariance is sometimes also referred to as a *potential algebra*. As we showed above in Eq. (45), the Hamiltonian in this case is given by J_+J_- , which commutes with J_3. All states of the representation, characterized by the eigenvalues of J_3, correspond to the same energy.

In an alternate version of SGA, the Hamiltonian is a linear combination of the generators of the algebra, and its eigenstates are directly related to the states of the representation. Analogous to the action of $a_\pm$ operators for the harmonic oscillator, one can connect eigenstates of the Hamiltonian by ladder operators to obtain the energy levels and wave functions by going up a single potential ladder. The first such algebra was found by Pauli and Fock [17] for the hydrogen atom. An example for a well known quantum mechanical problem appears in Ref. [16]. In Ref. [7], for two explicit examples, Balantekin has shown that the supersymmetric Hamiltonian is indeed part of an underlying SGA (it is a linear combination of two diagonal generators of the algebra) and thus the energy spectra can be determined algebraically. For the three-dimensional harmonic oscillator with spin-orbit coupling, Eq. (8), discussed earlier in the paper, Balantekin [7] defines three operators:

$$K_+ = \frac{1}{2}\sum_i^3 a_i^\dagger\, a_i^\dagger, \quad K_- = \frac{1}{2}\sum_i^3 a_i\, a_i, \quad \text{and} \quad K_0 = \frac{1}{2}\sum_i^3 \left(a_i^\dagger\, a_i + \frac{1}{2}\right).$$

These operators obey $[K_0, K_\pm] = \pm K_\pm$, $[K_+, K_-] = -2K_0$; i.e., they define an $sp(2) \sim so(2,1)$ algebra. In addition, he also defines five operators,

$$Y = \frac{1}{2}\left(\frac{3}{2} + \vec{\sigma}\cdot\vec{L}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$V_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_i a_i^\dagger \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_i a_i \\ 0 & 0 \end{pmatrix},$$

$$W_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \sigma_i a_i^\dagger & 0 \end{pmatrix} \quad \text{and} \quad W_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \sigma_i a_i & 0 \end{pmatrix}.$$

These operators obey:

$$[K_0, V_\pm] = \pm\tfrac{1}{2}V_\pm, \quad [K_0, W_\pm] = \pm\tfrac{1}{2}W_\pm,$$
$$[K_+\pm, V_\pm] = [K_-\pm, V_\pm] = 0, \quad [K_\pm, V_\mp] = \mp V_\pm, \quad [K_\pm, W_\mp] = \mp W_\pm,$$
$$[Y, V_\pm] = \tfrac{1}{2}V_\pm, \quad [Y, W_\pm] = -\tfrac{1}{2}W_\pm, \quad [Y, K_\pm] = [Y, K_0] = 0,$$

$$\{V_\pm, V_\pm\} = \{W_\pm, W_\pm\} = 0, \quad \{V_+, V_-\} = \{W_+, W_-\} = 0, \quad \{V_\pm, W_\pm\} = K_\pm,$$
$$\{V_-, W_+\} = K_0 + Y, \quad \text{and} \quad \{V_+, W_-\} = K_0 - Y.$$

I.e., they define the $osp(2/2)$ superalgebra [7]. Balantekin also shows that the Hamiltonian given by Eq. (8) can be written as

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = 2(K_0 + Y).$$

I.e., the above defined $osp(2/2)$ is indeed a spectrum generating algebra for the Hamiltonian. For a more detailed study of this type of SGA, specifically in the context of SUSYQM, we refer the reader again to Ref. [7].

We can see that it took much ingenuity and hard work to identify this SGA. Unlike the SGA derived from shape invariance, there is no algorithm available, as far as we know, to determine the spectrum generating algebra of the latter kind for all solvable potentials. Instead, it is found for each case separately.

7 Broken supersymmetry

In this section we study shape invariant potentials in which the parameters have values such that the supersymmetry is spontaneously broken. For these systems, the usual shape invariance procedure does not enable one to determine the spectra, since there is no zero energy eigenstate. The first solution to the broken SUSY problem was provided by Dutt et al. [11] for the three dimensional harmonic oscillator and the Pöschl-Teller I and II potentials. These are the only known exactly solvable broken SUSY potentials that hold bound states.

Gangopadhyaya et al. subsequently showed that there exists a two-step shape invariance that renders them exactly solvable [27]. The first step converts the initial broken SUSY potential into an unbroken one. The unbroken potential can then be solved in a second step using the standard procedure described earlier. This approach leads in a natural manner to the underlying potential algebra of broken SUSY potentials.

We may demonstrate the two-step shape invariance method by explicitly considering a generalized Pöschl-Teller I potential. We will show the types of shape invariance it possesses, and then exploit these to determine the eigenspectrum. The same method is then extended to the three dimensional harmonic oscillator and generalized Pöschl-Teller II potentials. Finally, we study the potential algebras underlying these systems, and generate their spectra by algebraic means.

7.1 Shape Invariant Potentials with Broken SUSY

Consider the generalized Pöschl-Teller I superpotential

$$W(x, A, B) = A \tan x - B \cot x; \quad 0 < x < \pi/2. \tag{59}$$

The supersymmetric partner potentials are given by

$$V_-(x,A,B) = A(A-1)\sec^2 x + B(B-1)\mathrm{cosec}^2 x - (A+B)^2;$$
$$V_+(x,A,B) = A(A+1)\sec^2 x + B(B+1)\mathrm{cosec}^2 x - (A+B)^2. \tag{60}$$

The (unnormalized) ground state wave function $\psi_0^{(-)}(x,A,B) \equiv \exp\left(-\int^x W(y,A,B)\,dy\right)$ is given by $\psi_0^{(-)}(x,A,B) = \cos^A x \sin^B x$. For $A>0, B>0$, the function $\psi_0^{(-)}(x,A,B)$ is normalizable, implying unbroken SUSY. Similarly, for $A<0, B<0$, the function $1/\psi_0^{(-)}(x,A,B)$ is normalizable, once again having unbroken SUSY, and the spectrum is obtainable from shape invariance. However, for $A>0, B<0$, $\psi_0^{(-)}(x,A,B)$ is not normalizable due to its divergent behavior at $x=0$ while its reciprocal $\frac{1}{\psi_0^{(-)}}$ is not normalizable due to its divergent behavior at $x=\frac{\pi}{2}$, and hence SUSY is broken. Likewise, for $A<0, B>0$, SUSY is broken.

Let us now focus on the case $A>0, B<0$. The eigenstates of $V_\pm(x,A,B)$ are related by

$$\psi_n^{(+)}(x,a_0) = A(x,a_0)\psi_n^{(-)}(x,a_0);\ \psi_n^{(-)}(x,a_0) = A^+(x,a_0)\psi_n^{(+)}(x,a_0);\ E_n^{(-)}(a_0) = E_n^{(+)}(a_0). \tag{61}$$

The potentials of Eq. (60) are shape invariant. In fact there are two possible transformation of parameters such that these two potentials exhibit shape invariance. The first is the conventional $A \to A+1,\ \ B \to B+1$. The shape invariance condition is given by

$$V_+(x,A,B) = V_-(x,A+1,B+1) + (A+B+2)^2 - (A+B)^2. \tag{62}$$

For B sufficiently large and negative, $B+1$ is also negative; thus the superpotential resulting from this change of parameters still falls in the broken SUSY category, and hence $E_0^{(-)}(a_0) \neq 0$ and therefore, even with shape invariance, we are not able to proceed further.

The second possible transformation is $A \to A+1,\ \ B \to -B$. The corresponding relationship between partner potentials is given by

$$V_+(x,A,B) = V_-(x,A+1,-B) + (A-B+1)^2 - (A+B)^2. \tag{63}$$

This change of parameters $A \to A+1,\ \ B \to -B$ leads to a system with unbroken SUSY, since the parameter B changes sign. Hence the ground state of the system with potential $V_-(x,A+1,-B)$ is guaranteed to be at zero energy. From Eq. (63), we see that potentials $V_+(x,A,B)$ and $V_-(x,A+1,-B)$ differ only by a constant. Hence

$$\psi_+(x,A,B) = \psi_-(x,A+1,-B);\ \ E_n^{(+)}(A,B) = E_n^{(-)}(A+1,-B) + (A+1-B)^2 - (A+B)^2.$$

Thus, since we know the spectrum of $H_-(x,A+1,-B)$ for unbroken SUSY [4, 5], we are able to determine the spectrum of $H_+(x,A,B)$ with broken SUSY. The results are

$$E_n^{(-)}(A+1,-B) = (A+1-B+2n)^2 - (A+1-B)^2. \tag{64}$$

When combined with Eqs. (61) and (63), Eq. (64) yields

$$\begin{aligned} E_n^{(-)}(A,B) &= (A+1-B+2n)^2-(A+B)^2; \\ \psi_n^{(-)}(y,A,B) &= (1+y)^{(1-A)/2}(1-y)^{B/2}P_n^{(B-1/2,1/2-A)}(y), \end{aligned} \tag{65}$$

where $y=\cos(2x)$ and $P_n^{(\alpha,\beta)}(y)$ are Jacobi polynomials, in agreement with the results of Ref. [11].

We have considered the Pöschl-Teller I potential in detail. Very similar analyses can be used for determining the eigenvalues and eigenstates of the three dimensional harmonic oscillator, as well as the generalized Pöschl-Teller II potential. We will now briefly describe the treatment of these potentials.

The three dimensional harmonic oscillator with broken SUSY is described by the superpotential

$$W(r,l,\omega)=\frac{1}{2}\,\omega r-\frac{l+1}{r};\quad l<-1. \tag{66}$$

The ground state function $\psi_0^{(-)}(r,l,\omega)=\exp\left(-\int^r W(r',l,\omega)\,dr'\right)=r^{l+1}e^{-\omega r^2}$ diverges near $r\to 0$ for $l<-1$ and hence corresponds to the case of broken SUSY. The supersymmetric partner potentials are

$$\begin{aligned} V_+(r,l,\omega) &= \frac{\omega^2 r^2}{4}+\frac{(l+1)(l+2)}{r^2}-\left(l+\frac{1}{2}\right)\omega \\ V_-(r,l,\omega) &= \frac{\omega^2 r^2}{4}+\frac{l(l+1)}{r^2}=\left(l+\frac{3}{2}\right)\omega, \end{aligned} \tag{67}$$

and they are related by

$$V_+(r,l,\omega)=V_-(r,l+1,\omega)+2\omega. \tag{68}$$

For $l<-1$, the potential $V_-(r,l+1,\omega)$ also lies in the realm of broken SUSY. Thus, the usual SUSYQM method again fails. However, there is another change of parameters $l\mapsto -l-2$ that also maintains shape invariance between these two partner potentials, namely,

$$V_+(r,l,\omega)=V_-(r,-l-2,\omega)-(2l+1)\omega. \tag{69}$$

Since $l<-1$, the potential $V_-(r,-l-2,\omega)$ has unbroken SUSY and has a zero energy ground state. Combining the results of Eqs. (67), (68) and (69), we get

$$E_n^{(+)}(l,\omega)=(2n-2l-1)\omega. \tag{70}$$

Our last example is the generalized Pöschl-Teller II potential described by

$$W(r,A,B)=A\tanh r-B\coth r;\qquad 0<r<\infty. \tag{71}$$

The function $\psi_0^{(-)}(r,A,B)\equiv\exp\left(-\int^r W(r',A,B)\,dr'\right)$ is given by

$$\psi_0^{(-)}(r,A,B)=\cosh^{-A}r\,\sinh^B r.$$

Here, for $A > 0$ and $B < 0$, neither $\psi_0^{(-)}(r, A, B)$ nor its inverse are normalizable, and hence we have a system with broken SUSY. The supersymmetric partner potentials are given by

$$\begin{aligned} V_+(r, A, B) &= -A(A-1)\text{sech}^2 r + B(B-1)\text{cosech}^2 r + (A+B)^2 \\ V_-(r, A, B) &= -A(A+1)\text{sech}^2 r + B(B+1)\text{cosech}^2 r + (A+B)^2. \end{aligned} \tag{72}$$

Here too we have two possible relations between parameters for these potentials to be shape invariant. They are

$$V_+(r, A, B) = V_-(r, A-1, B-1) + (A+B)^2 - (A+B-2)^2 \tag{73}$$

and

$$V_+(r, A, B) = V_-(r, A-1, -B) + (A+B)^2 - (A-B-1)^2. \tag{74}$$

As explained in the previous two examples, the first transformation does not lead to the determination of the spectrum since the new parameters $(A-1,\ B-1)$ still correspond to broken SUSY for sufficiently large positive value of A. However, in the second type of transformation, the new values of the parameters $(A-1,\ -B)$ lie in the domain of unbroken SUSY and hence the spectrum of $V_-(r, A-1, -B)$ can be determined by standard methods of supersymmetric quantum mechanics. The resulting spectrum is given by

$$E_n^{(-)}(A, B) = (A+B)^2 - (A-1-B-2n)^2.$$

7.2 Spectra of Broken SUSY Problems using Potential Algebras

The shape invariance based approach to broken supersymmetric potentials discussed above also naturally leads us to the underlying potential algebra, which in turn allows us to determine the spectrum by algebraic means [13-15]. As we have seen above, the algebraic approach closely mimics the SUSYQM approach, so we shall simply outline the procedure here. A full description appears in Ref. [27].

Analogous to the combination of the two step invariances, for broken SUSY we have to combine two different algebras. We define the operators $J_\pm, J_3$ as in Theorem 1, with their algebra corresponding to the spectrum of a given Hamiltonian. To connect the broken SUSY sector to the unbroken SUSY sector, we introduce three additional operators $K^+,\ K^-,$ and K_3 with an algebra that emulates the shape invariance condition for broken SUSY, namely, inversion rather than translation — i.e., $A \mapsto -A$ rather than $A \mapsto A \pm 1$. This has the effect of transforming the broken SUSY problem into an unbroken SUSY problem.

8 Conclusions

Starting from Pauli and Fock, many authors have shown an association of algebraic structures with exactly solvable systems of quantum mechanics. In the context of supersymmetric quantum mechanics, all known solvable potentials have been shown to be

shape invariant. In this paper, we have reviewed the work connecting shape invariance to a spectrum generating algebra known as a potential algebra. We have described a unified approach to demonstrating the existence of an underlying algebra in all solvable systems, thus providing a "deeper" reason for their solvability. We have developed a theorem which provides a schema for researchers to investigate the algebra of shape invariant potentials. We have provided worked examples so that newcomers to the field can see how the theory is employed to produce concrete results.

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