Gravitational slingshot

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The slingshot effect is an intriguing phenomenon that has been used effectively by NASA to send spacecraft to outer edges of the solar system. This phenomenon can be satisfactorily explained by Newtonian physics. However, if it is presented as a problem involving four-momentum conservation, the methods of relativistic kinematics easily lead to the conditions necessary for an accelerating as well as a retarding scenario. This problem provides an example that showcases the frequent utility of relativistic methods to analyze problems of Newtonian mechanics.

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I. INTRODUCTION

The slingshot effect may seem puzzling in light of the conservative nature of gravity. If the initial conditions are right, as a craft whizzes past a planet, it picks up kinetic energy from the gravitational interaction. NASA routinely uses this effect to provide boosts to their spacecraft to explore the solar system.[1,2] Voyager I and II used the boost provided by Jupiter to reach Uranus and Neptune. Cassini will utilize four such assists on its way to reach Saturn in 2004. The question is how do these boosts work. The major source of the confusion is that an elastic collision is supposed to conserve the kinetic energy of the system. Because we do not expect, albeit erroneously, the planet to have any appreciable change in kinetic energy, we find it difficult to understand why there should be any change in kinetic energy of the craft either.

In this paper, we show how this mechanism works. In particular, we determine the conditions necessary for achieving a change in the speed of the craft as it goes past a planet. There are several levels at which this phenomenon can be analyzed. Reference 3 gives an intuitive account for this effect, and Refs. 4–7 explain the quantitative aspects using Newton's laws of motion. We employ the method of relativistic kinematics. The use of special relativity theory to derive a Newtonian result may seem like overkill. However, this approach exploits the unification of linear momentum and energy between a planet-centered frame of reference and sun-centered frame, and the result is derived in one step. Thus the major reason for a relativistic approach is its elegance and simplicity. This derivation also exemplifies how the formalism of relativistic mechanics can be used to derive information about the Newtonian world as an approximation. It provides a simple and compact formalism for comparing momentum and energy in two inertial frames using simple linear transformations.

II. ANALYSIS

Let us assume the planet to be moving along the x axis with a velocity $\mathbf{V}=V\hat{e}_x$ as seen from a sun-centered frame. Due to the radial nature of the gravitational force and consequent conservation of angular momentum, the motion of the planet is confined to a plane, the ecliptic plane. In this plane, which we will denote as the xy plane, the craft scatters off the planet whose mass is enormous with respect to the craft (about 10^{20} times or more). Due to the ratio of the planet's mass to the craft's mass, a planet-centered frame, henceforth called the P-frame, can be assumed to be the center of mass frame of the two objects. In the P-frame, the interaction between the craft and the planet can be viewed as the scattering of the craft off a fixed center, and the kinetic energy of the craft will be conserved.

Let the initial speed of the craft relative to the planet be u. Its final speed, with respect to the planet, will be the same. The corresponding momentum four-vectors of the craft in the P-frame before and after a collision (interaction) are denoted by

$$\mathbf{p}_{1,\mathrm{P}}=\begin{pmatrix} mu\gamma_u\cos\theta_1 \\ mu\gamma_u\sin\theta_1 \\ 0 \\ mc\gamma_u \end{pmatrix} \quad \text{(before a collision)}, \tag{1a}$$

$$\mathbf{p}_{2,\mathrm{P}}=\begin{pmatrix} mu\gamma_u\cos\theta_2 \\ mu\gamma_u\sin\theta_2 \\ 0 \\ mc\gamma_u \end{pmatrix} \quad \text{(after a collision)}, \tag{1b}$$

where $\gamma_u=1/\sqrt{1-u^2/c^2}$. The angles θ_1 and θ_2 are the angles between the x axis of the P-frame and the velocities of the craft before and after the interaction, respectively. To ascertain whether there is a boost due to the interaction, we will have to determine these velocities in a frame that is at rest with respect to the sun, that is, in the S-frame. In this frame, the velocity of the planet is given by $\mathbf{V}=V\hat{e}_x$. To find the component of momenta in the S-frame, we apply the following Lorentz matrix, L_V, to the energy–momentum vectors given in Eq. (1):

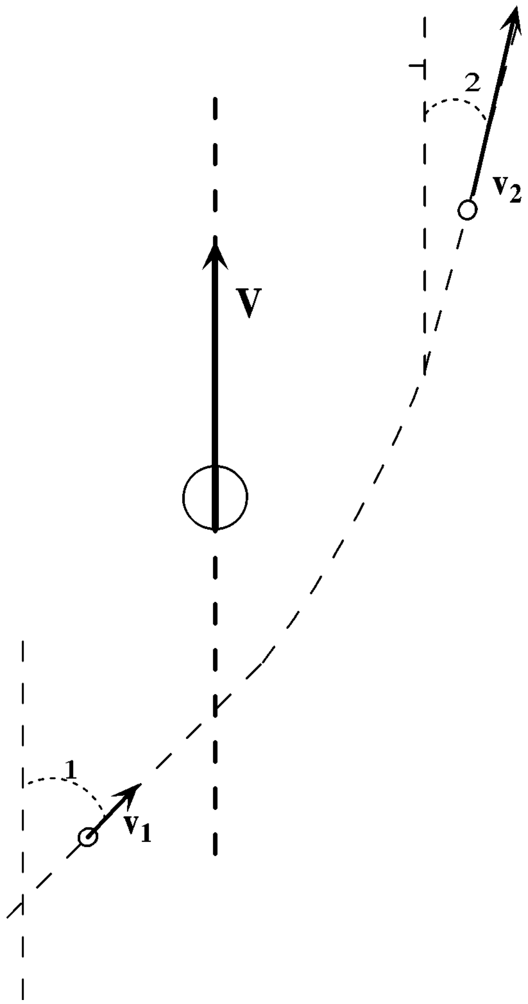

Fig. 1. This approach leads to increase of kinetic energy for the spacecraft.

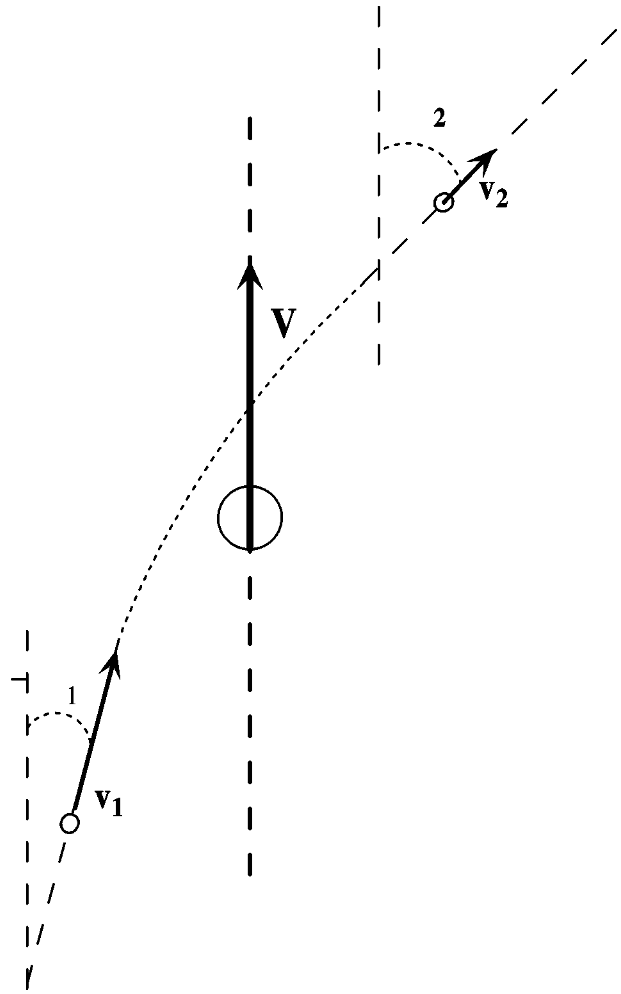

Fig. 2. This approach results in a decrease of kinetic energy for the spacecraft.

$$L_V = \begin{vmatrix} \gamma_V & 0 & 0 & V\gamma_V/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ V\gamma_V/c & 0 & 0 & \gamma_V \end{vmatrix}, \tag{2}$$

which leads to

$$\mathbf{p}_{1,\mathrm{S}} = L_V \cdot \mathbf{p}_{1,\mathrm{P}} = \begin{pmatrix} m v_1 \cos\phi_1 \gamma_{v_1} \\ m v_1 \sin\phi_1 \gamma_{v_1} \\ 0 \\ m c \gamma_{v_1} \end{pmatrix}$$

$$= \begin{pmatrix} m(u\cos\theta_1 + V)\gamma_u\gamma_V \\ m u \sin\theta_1 \gamma_u \\ 0 \\ m\left(\frac{uV}{c}\cos\theta_1 + c\right)\gamma_u\gamma_V \end{pmatrix}, \tag{3}$$

where v_1 is the nonrelativistic speed of the craft and ϕ_1 is the angle between its velocity and x axis in the S-frame before the interaction with the planet. The factors γ_{v_1} and γ_V are $1/\sqrt{1-v_1^2/c^2}$ and $1/\sqrt{1-V^2/c^2}$, respectively. Analogous relations hold for the components of energy–momentum in the two frames. If we divide the second component by the first component, we obtain

$$\tan\phi_i = \frac{\sqrt{1-V^2/c^2}\, u \sin\theta_i}{u\cos\theta_i + V}, \tag{4}$$

which determines the before and after angles, ϕ_1 and ϕ_2, between the velocity of the craft and the x axis in the S-frame in terms of the P-frame angles, θ_1 and θ_2, respectively. In the classical limit, we find the expected result:

$$\tan\phi_i = \frac{u\sin\theta_i}{u\cos\theta_i + V} \quad (\text{classical limit}). \tag{5}$$

The fourth component of $\mathbf{p}_{1,\mathrm{S}}$, the energy divided by c, reveals the relativistic kinetic energy of the craft in the S-frame before interaction,

$$E - E_0 = mc^2\gamma_{v_1} - mc^2 = mc^2\left(\left(\frac{uV}{c^2}\cos\,\theta_1 + 1\right)\gamma_u\gamma_V - 1\right). \tag{6}$$

Similarly, the relativistic kinetic energy of the craft in the S-frame after the interaction is

$$mc^2\left(\left(\frac{uV}{c^2}\cos\ \theta_2+1\right)\gamma_u\gamma_V-1\right).$$

Thus the change in kinetic energy of the craft due to the interaction with the planet is

$$mc^2(\gamma_{v_2}-\ \gamma_{v_1})=\frac{mc^2\frac{uV}{c^2}(\cos\theta_2-\cos\theta_1)}{\sqrt{1-u^2/c^2}\sqrt{1-V^2/c^2}}$$
$$\approx muV(\cos\theta_2-\cos\theta_1). \quad (7)$$

Thus, to generate an increase of the kinetic energy of the craft, it would have to approach the planet in such way that $\theta_2<\theta_1$, that is, $\cos\theta_2>\cos\theta_1$. This situation is shown in Fig. 1. On the other hand, if a retardation is desired, the angles will have to be chosen such that $\theta_1<\theta_2$, that is, $\cos\theta_1>\cos\theta_2$ (see Fig. 2).

III. CONCLUSION

We have analyzed the kinematics of the gravitational slingshot effect experienced by spacecraft as they go past massive planets. The increase in the kinetic energy of the craft is at the expense of a decrease in kinetic energy of the planet, albeit by a very small fraction due to its enormous inertial mass. Both momentum and kinetic energy play very important roles in this analysis. Relativistic analysis tells us to transform both physical quantities simultaneously as one four-dimensional vector.

We showed that if a craft approaches the planet as shown in Fig. 1, that is, $\theta_2<\theta_1$, it results in an increase of the kinetic energy of the craft. On the other hand, if we desire a retardation, the craft would have to approach the planet from in front as seen in Fig. 2.[8] If one needs successive boosts to reach far away planets, one would need some special temporal windows to achieve this when the planets are very specially aligned.

The formalism of relativistic kinematics has been used mainly for its elegance and simplicity. It is also intended to show that, in some cases, the relativistic method is shorter and more compact than its non-relativistic counterpart.

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[1]A list of planetary missions that have used gravity assist can be found at ⟨http://bbs.cshs.tcc.edu.tw/geology/star/chinese/nineplanets/spacecraft.html⟩.

[2]"Voyager: Journey to the Outer Planets," NASA, JPL SP 43-39. NASA-JPL-COML, Los Angeles, CA, Contract No. NAS7-100, Jet Propulsion Laboratory, Pasadena, CA, p. 2.; R. O. Fimmel, J. Van Allen, and E. Burgess, "Pioneer: First to Jupiter, Saturn and Beyond," NASA SP 446 (NASA, Washington, DC, 1980).

[3]L. Epstein, "Pioneer Jupiter whiplash," Phys. Teach. **11**, 299–300 (1973).

[4]Albert A. Bartlett and Charles W. Hord, "The slingshot effect: Explanation and analogies," Phys. Teach. **23**, 466–473 (1985).

[5]James A. Van Allen, "Gravitational assist in celestial mechanics: A tutorial," Am. J. Phys. **71**, 448–451 (2003).

[6]R. C. Johnson, "The slingshot effect," ⟨http://www.dur.ac.uk/bob.johnson/SL/⟩.

[7]L. Bogan, "Satellite orbits gravitational assist from planets," ⟨http://www.go.ednet.ns.ca/larry/orbits/gravasst/gravasst.html⟩.

[8]This slowing of the craft occurs in the Sun frame and would not be useful for the purpose of landing on the planet.