Algebraic shape invariant models

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Abstract

Motivated by the shape invariance condition in supersymmetric quantum mechanics, we develop an algebraic framework for shape invariant Hamiltonians with a general change of parameters. This approach involves nonlinear generalizations of Lie algebras. Our work extends previous results showing the equivalence of shape invariant potentials involving a translational change of parameters with a standard $SO(2,1)$ potential algebra for Natanzon type potentials. © 1998 Published by Elsevier Science B.V.

In recent years, there has been considerable interest in studying exactly solvable quantum mechanical problems using algebraic approaches [1-9]. In this respect, supersymmetric quantum mechanics (SUSYQM) has been found to be an elegant and useful prescription for obtaining closed analytic expressions both for the energy eigenvalues and eigenfunctions for a large class of one-dimensional (or spherically symmetric three-dimensional) problems. The main ingredients in SUSYQM are the supersymmetric partner Hamiltonians $H_- \equiv A^\dagger A$ and $H_+ \equiv AA^\dagger$. The A and $A^\dagger$ operators used in this factoriza-

tion are expressed in terms of the real superpotential W as follows,

$$A(x,a) = \frac{d}{dx} + W(x,a),$$
$$A^\dagger(x,a) = -\frac{d}{dx} + W(x,a). \qquad (1)$$

Here, a is a parameter (or a set of parameters), which will play an important role in the approach of this paper. An interesting feature of SUSYQM is that for a shape invariant system [1-3], i.e. a system satisfying

$$H_+(x,a_0) = H_-(x,a_1) + R(a_0),$$
$$a_1 = f(a_0), \qquad (2)$$

the entire spectrum can be determined algebraically without ever referring to underlying differential equations.

Recently, it has been shown that shape invariant potentials in which the change of parameters $a_1 = f(a_0)$

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is of translational type $a_1 = a_0 + k$, k = const, possess a $SO(2, 1)$ potential algebra and are solvable by group theoretical techniques [4–7]. Thus, for such potentials, there appear to be two seemingly independent algebraic methods, SUSYQM and potential algebras, for obtaining the complete spectra. One may naturally ask the question whether there is any connection between the two methods. Indeed, the equivalence between the two algebraic approaches has been established [6]. More specifically, for all shape invariant potentials of the Natanzon type [10], the underlying potential algebra is $SO(2, 1)$. Other similar approaches are discussed in Refs. [4,11]

The purpose of this Letter is to examine whether such an equivalence is also valid for a larger class of shape invariant potentials. We consider a much more general class of change of parameters, which contains both translations as well as scalings. In particular, for shape invariant potentials generated by a pure scaling ansatz [12] $a_1 = qa_0$, q = const ($0 < q < 1$), we find that the associated potential algebra is a nonlinear deformation of $SU(2)$. In fact, coordinate representations of these nonlinear algebras are an interesting area of research in their own right from a mathematical point of view [13]. We also show that the energy spectrum resulting from this potential algebra agrees with the results of SUSYQM.

To begin the construction of the operator algebra, let us express Eq. (2) in terms of A and $A^\dagger$,

$$A(x, a_0)A^\dagger(x, a_0) - A^\dagger(x, a_1)A(x, a_1) = R(a_0) \,. \tag{3}$$

This relation resembles a commutator structure. To obtain a closed $SU(2)$-like algebra, we introduce an auxiliary variable ϕ and define

$$J_+ = Q(\phi)A^\dagger(x, \chi(i\partial_\phi)) \,,$$
$$J_- = A(x, \chi(i\partial_\phi))Q^*(\phi) \,, \tag{4}$$

where $Q(\phi)$ is a function to be determined and χ is an arbitrary, real function. The operator $A(x, \chi(i\partial_\phi))$ is given by Eq. (1) with the substitution

$$a \to \chi(i\partial_\phi) \,. \tag{5}$$

From equation (4), one obtains

$$[J_+, J_-]$$
$$= Q(\phi)A^\dagger(x, \chi(i\partial_\phi))A(x, \chi(i\partial_\phi))Q^*(\phi)$$
$$- A(x, \chi(i\partial_\phi))Q^*(\phi)Q(\phi)A^\dagger(x, \chi(i\partial_\phi)) \,. \tag{6}$$

Choosing $Q(\phi) = e^{ip\phi}$, where p is an arbitrary real constant, Eq. (6) can be easily cast into the following form,

$$[J_+, J_-] = -\{A(x, \chi(i\partial_\phi))A^\dagger(x, \chi(i\partial_\phi))$$
$$- A^\dagger(x, \chi(i\partial_\phi + p))A(x, \chi(i\partial_\phi + p))\} \,. \tag{7}$$

In analogy with Eq. (3), we impose the algebraic shape invariance condition

$$A(x, \chi(i\partial_\phi))A^\dagger(x, \chi(i\partial_\phi)) - A^\dagger(x, \chi(i\partial_\phi + p))$$
$$\times A(x, \chi(i\partial_\phi + p)) = R(\chi(i\partial_\phi)) \,. \tag{8}$$

Note that the quantity J_+J_- corresponds to the Hamiltonian

$$H_-(x, i\partial_\phi + p) = A^\dagger(x, \chi(i\partial_\phi + p))$$
$$\times A(x, \chi(i\partial_\phi + p)) \,. \tag{9}$$

Identifying

$$J_3 = -\frac{i\partial_\phi}{p} \,, \tag{10}$$

we obtain a deformed $SO(2, 1)$ Lie algebra whose generators J_+, J_- and J_3 satisfy

$$[J_3, J_\pm] = \pm J_\pm \,,$$
$$[J_+, J_-] = \xi(J_3) \,. \tag{11}$$

Here $\xi(J_3) \equiv -R(\chi(i\partial_\phi))$. The algebraic shape invariance condition (3) is the main tool of our analysis. If we identify

$$a_0 \to \chi(i\partial_\phi) \,, \quad a_1 \to \chi(i\partial_\phi + p) \,, \tag{12}$$

then depending on the choice of the χ function in Eq. (12), we have (1) translational models: $a_1 = a_0 + p$ for $\chi(z) = z$, (2) scaling models: $a_1 = e^p a_0 \equiv qa_0$ for $\chi(z) = e^z$.

One may note that in the translational reparametrization ($\chi(z) = z$) the commutation relations (11) describe a standard, undeformed $SO(2, 1)$ Lie algebra with $\xi(J_3) = -2J_3$, as found in Ref. [6]. Other
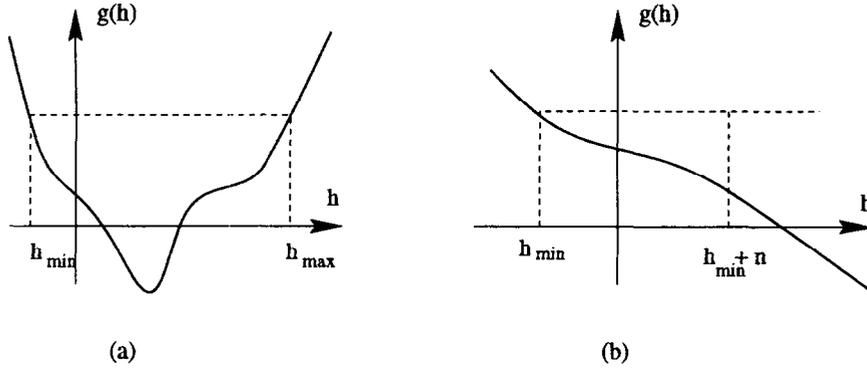
$$(a) \qquad\qquad (b)$$

Fig. 1. Generic behaviors of $g(h)$.

changes of parameters follow from more complicated choices for $\chi(z)$. For example, if one takes $\chi(z) = e^{e^z}$, one gets the change of parameters $a_1 = a_0^2$.

As a first example of scaling, we choose $R(a_0) = R_1 a_0$. This choice generates self-similar potentials studied in Refs. [12,14]. In this case, Eqs. (11) become

$$[J_3, J_\pm] = \pm J_\pm \,,$$
$$[J_+, J_-] = \xi(J_3) \equiv -R_1 \exp(-pJ_3) \,. \qquad (13)$$

Thus, we see that the resulting potential algebra is no longer given by a $SO(2,1)$ Lie algebra. Instead, it is described by a deformed $SO(2,1)$. To find the energy spectrum of the Hamiltonian H_- of Eq. (9), we first construct the unitary representations of this deformed Lie algebra [15]. We define a function $g(J_3)$ such that

$$\xi(J_3) = g(J_3) - g(J_3 - 1) \,. \qquad (14)$$

The function $g(J_3)$ is unique up to an additive unit periodic function. The Casimir of this algebra is then given $C_2 = J_- J_+ + g(J_3)$. It is known that in a basis in which J_3 and C_2 are diagonal, J_+ and J_- play the role of raising and lowering operators, respectively. Operating on an arbitrary state $|h\rangle$, we have

$$J_3|h\rangle = h|h\rangle \,,$$
$$J_-|h\rangle = a(h)|h - 1\rangle \,,$$
$$J_+|h\rangle = a^*(h + 1)|h + 1\rangle \,. \qquad (15)$$

Using Eqs. (11) and (15) we obtain

$$|a(h)|^2 - |a(h + 1)|^2 = g(h) - g(h - 1) \,. \qquad (16)$$

The profile of $g(h)$ determines the dimension of the unitary representation. To illustrate how this mechanism works, let us consider the two cases presented in Fig. 1. One obtains finite dimensional representations Fig. 1a, by starting from a point on the $g(h)$ versus h graph corresponding to $h = h_{min}$, and moving in integer steps parallel to the h-axis till the point corresponding to $h = h_{max}$. At the end points, $a(h_{min}) = a(h_{max} + 1) = 0$, and we get a finite representation. (This is the case of $SU(2)$ for example, where $g(h)$ is given by the parabola $h(h + 1)$.) If $g(h)$ is decreasing monotonically, Fig. 1b, there exists only one end point at $h = h_{min}$. Starting from h_{min}, the value of h can be increased in integer steps till infinity. In this case we have an infinite dimensional representation. As in the finite case, h_{min} labels the representation. The difference is that here h_{min} takes continuous values. Similar arguments apply for a monotonically increasing function $g(h)$. A more detailed discussion of these techniques is given in Ref. [15].

In our case, from Eqs. (13) and (14) one gets

$$g(h) = \frac{R_1}{e^p - 1} e^{-ph} = -\frac{R_1}{1 - q} q^{-h} \,,$$
$$q = e^p \,, \qquad (17)$$

represented graphically in Fig. 2. Note that for scaling problems [12], one requires $0 < q < 1$, which leads to $p < 0$. From the monotonically decreasing profile of the function $g(h)$, it follows that the unitary representations of this algebra are infinite dimensional. If we label the lowest weight state of the operator J_3 by h_{min}, then $a(h_{min}) = 0$. Without loss of generality we can choose the coefficients $a(h)$ to be real. Then

one obtains from (16) for an arbitrary $h = h_{min} + n$, $n = 0, 1, 2, \ldots,$

$$a^2(h) = g(h - n - 1) - g(h - 1) = R_1 \frac{q^n - 1}{q - 1} q^{1-h},$$
(18)

where $q = e^p$. The spectrum of the Hamiltonian $H_-(x, a_0)$ is given by

$$H_-|h\rangle = a^2(h)|h\rangle = R_1 \frac{q^n - 1}{q - 1} q^{1-h}|h\rangle.$$
(19)

Therefore, the eigenenergy is

$$E_n(h) = \alpha(h) \frac{q^n - 1}{q - 1}, \quad \alpha(h) \equiv R_1 q^{1-h}.$$
(20)

Thus, the eigenvalue h of the operator J_3 plays the role of a parameter for the theory. To compare the spectrum of this Hamiltonian obtained in Eq. (19) with those obtained from SUSYQM [12], we go to the x-representation. Here $|h\rangle \propto e^{iph\phi}\psi_{h_{min},n}(x)$ and hence, the Schrödinger equation for the Hamiltonian H_- reads

$$\left\{ -\frac{d^2}{dx^2} + W^2(x, e^{i\partial_\phi + p}) - W'(x, e^{i\partial_\phi + p}) - E \right\}$$
$$\times e^{ip\phi h}\psi_{h_{min},n}(x) = 0,$$

or

$$\left\{ -\frac{d^2}{dx^2} + W^2(x, \alpha) - W'(x, \alpha) - E \right\}$$
$$\times \psi_{h_{min},n}(x) = 0,$$
(21)

which is exactly the Schrödinger equation for the scaling model in SUSYQM. Hence, the energy E is given by Eq. (20) for $h = $ const. In Fig. 2, we depict the elegant correspondence that exists between potential algebra and supersymmetric quantum mechanics for shape invariant potentials. Let us fix $h = h_{min} + n$ to a given value, say h_0. For $n = 0$ we have a representation of our deformed $SO(2, 1)$ algebra labeled by $h_0 \equiv h_{min}$. We denote the corresponding potential from the Schrödinger equation of this Hamiltonian by V_0. From Eq. (20) the energy $E_0(h_0) = 0$ as we should expect for the ground state in SUSYQM. For $n = 1$, we obtain the first excited state for V_0 at $E_1(h_0) = \alpha(h_0)$. Note that we can write $h_0 = (h_{min} - 1) + 1 \equiv h_1 + 1$. Construct now a new representation labeled by $h_1 \equiv$
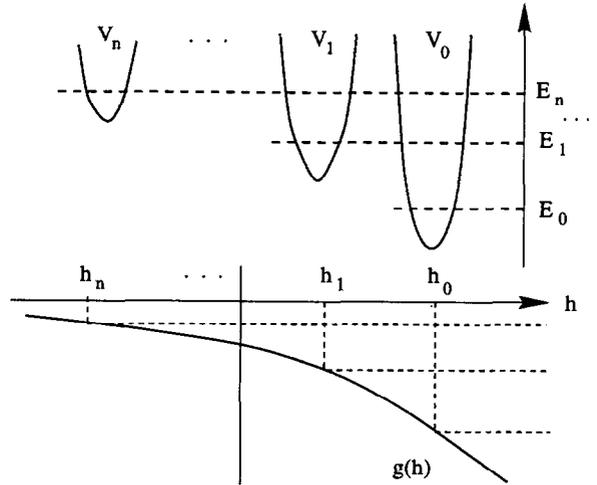

Fig. 2. Schematic algebra showing the correspondence between potential algebra and SUSYQM with scaling type reparametrization.

$h_0 - 1$. The corresponding potential V_1 has its ground state energy, at the same level with the first excited state of V_0. But this is the well-known property of the supersymmetric partner potentials. One can now iterate this procedure. After n steps, we build a representation labeled by h_n where $h_n = h_0 - n$, and the ground state energy of the corresponding potential V_n will be exactly the n-excited state energy of V_0.

For a more general case, we assume $R(a_0) = \sum_{j=1}^{\infty} R_j a_0^j$. In this case

$$g(h) = \sum_{j=1}^{\infty} \frac{R_j}{1 - e^{jp}} e^{-jph},$$
(22)

and therefore one gets

$$a^2(h) = g(h - n - 1) - g(h - 1)$$
$$= \sum_{j=1}^{\infty} \alpha_j(h) \frac{1 - q^{jn}}{1 - q^j},$$
(23)

where $\alpha_j(h) = R_j e^{-j(h-1)}$. This result agrees with those obtained in Ref. [12].

In conclusion, in this paper we have developed an algebraic approach involving nonlinear generalizations of Lie algebras for treating general shape invariant Hamiltonians.

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