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Analysis of inverse-square potentials using supersymmetric quantum mechanics

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Abstract. We analyse transition potentials, i.e. potentials exhibiting limiting inverse-square behaviour $V(r) \overset{r\to 0}{\to} \alpha r^{-2}$ in non-relativistic quantum mechanics using the techniques of supersymmetry. For the range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$, the eigenvalue problem becomes ill defined (since it is not possible to choose a unique eigenfunction based on square integrability and boundary conditions). It is shown that supersymmetric quantum mechanics (SUSYQM) provides a natural prescription for a unique determination of the spectrum. Interestingly, our SUSYQM-based approach picks out the same 'less singular' wavefunctions as the conventional approach and thus provides a simple justification for the usual practice in the literature. Two examples (the Pöschl–Teller II potential and a two-anyon system on the plane) have been worked out for illustrative purposes.

Transition potentials in non-relativistic quantum mechanics are defined by

$$\lim_{r\to 0} r^2 V(r) = \alpha$$

with a finite non-zero α [1]. For $\alpha \geqslant \frac{3}{4}$, the eigenvalue problem is well defined and can be solved by conventional means. For $\alpha < -\frac{1}{4}$, both independent solutions of the Schrödinger differential equation are square integrable at the origin and there is no mechanism available to select any specific linear combination.

However, for the intermediate range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$, transition potentials exhibit a very interesting behaviour. This range corresponds to the so-called 'limit-circle' case in the literature [1] and one has to specify another real number $c = \lim_{r\to 0}(\psi'(r)/\psi(r))$ in order to make the Hamiltonian formally self-adjoint. Here, the requirement of square integrability is not sufficient to determine the eigenvalues. Even with the stronger condition that the wavefunction must vanish at the origin ($c = \infty$), the non-uniqueness still persists, albeit for a smaller range of α given by $-\frac{1}{4} \leqslant \alpha < 0$. For values of α in this interval, eigenvalues are not well defined in the absence of further assumptions. This lack of uniqueness arises from the fact that both linearly-independent solutions of the Schrödinger equation are well defined near the origin and the condition of square integrability does not help us in discarding one of them. In such cases, it is customary [1] to force the coefficient of the term with the smaller power of r to vanish. This conventional approach of retaining the

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'less singular' wavefunction then leads to the determination of eigenvalues and well-defined eigenfunctions. Frank *et al* [1], in their comprehensive study, justified the above choice through a regularization procedure. Specifically, the potential is first made regular in a small neighbourhood of the singular point with a radius γ. After matching the solutions at the boundary and taking the limit $\gamma \to 0$, the less singular wavefunction is selected.

For any spherically-symmetric potential in three dimensions, an α/r^2 term arises from the angular-momentum term in the Hamiltonian. The requirement of single valuedness constrains the coefficient of α to be positive. However, in two-dimensional quantum-mechanical systems, angular momentum can take any real value and, thus, the problem of indeterminacy is relevant to the quantum mechanics of anyons [2]. The problem is also relevant for many known physically-interesting one-dimensional potentials (Rosen–Morse, Eckart, Pöschl–Teller, etc) which have an r^{-2} behaviour at the origin.

In this paper, we provide an alternative way for determining eigenvalues in the critical range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$. Our method is based on the supersymmetric approach to quantum mechanics [3]. The key idea is that in situations in which the eigenfunctions of a potential V_- are not unambiguously determined, the supersymmetric partner potential V_+ has no such problem. Thus, solving for V_+ first and then using the degeneracy relation, one can solve the eigenvalue problem for the potential V_-. Interestingly, we find that our approach leads to the same answer as the one stated in [1]. Hence, this paper provides an alternative justification for the prescription of choosing the 'less singular' solution, which obviously works.

To be complete, we give a brief review of supersymmetric quantum mechanics (SUSYQM). For a detailed description of SUSYQM, we refer the reader to [3]. We use examples of the Pöschl–Teller potential and that of a two-anyon system to describe how SUSYQM provides a method for resolving the indeterminacy mentioned earlier.

SUSYQM is characterized by a superpotential W and a pair of linear operators A and $A^\dagger$

$$A = \frac{\mathrm{d}}{\mathrm{d}r} + W(r) \qquad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}r} + W(r). \tag{1}$$

Combining these operators, we can define two Hamiltonians

$$H_- = A^\dagger A = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_-(r) \tag{2}$$

$$H_+ = AA^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_+(r) \tag{3}$$

$$V_\pm(r) = W^2(r) \pm \frac{\mathrm{d}W(r)}{\mathrm{d}r}. \tag{4}$$

We have set $\hbar = 2m = 1$. The potentials V_+ and V_- are called supersymmetric partner potentials. The eigenstates of the Hamiltonians H_- and H_+ are $\psi_n^{(-)}$ and $\psi_n^{(+)}$, respectively. The $\psi_n^{(\pm)}$ satisfy the eigenvalue equations

$$H_-\psi_n^{(-)} = E_n^{(-)}\psi_n^{(-)} \qquad H_+\psi_n^{(+)} = E_n^{(+)}\psi_n^{(+)}. \tag{5}$$

If the ground state of H_- has zero energy, i.e. $E_0^{(-)} = 0$, then supersymmetry is said to be unbroken and one has $A\psi_0^{(-)} = 0$. It then follows from equation (1) that

$$\psi_0^{(-)} = \exp\left[-\int^r W(r')\,\mathrm{d}r'\right]. \tag{6}$$

For unbroken supersymmetry, one needs $\psi_0^{(-)}$ or $1/\psi_0^{(-)}$ to be an acceptable wavefunction, i.e. it must be quadratically integrable and satisfy correct boundary conditions. For a finite domain, the wavefunction must vanish at the end points. For a normalizable well-defined $\psi_0^{(-)}$, one gets the energy-degeneracy relation

$$E_{n+1}^{(-)} = E_n^{(+)}. \tag{7}$$

The corresponding eigenfunctions of H_- and H_+ are related by

$$\psi_{n+1}^{(-)} = A^\dagger \psi_n^{(+)}. \tag{8}$$

The applicability of SUSYQM to lift the ambiguity in the determination of eigenvalues and eigenfunctions that plague transition potentials can be appreciated from the following discussion. If the superpotential $W(r)$ is given by $-(l+1)/r$ near $r = 0$, then potentials $V_-(r)$ and $V_+(r)$ are of the form $l(l+1)/r^2$ and $(l+1)(l+2)/r^2$ near $r = 0$. Their wavefunctions are given by linear combinations $[c_1 r^{-l}(1 + \mathcal{O}(r)) + c_2 r^{l+1}(1 + \mathcal{O}(r))]$ and $[c_1' r^{-l-1}(1+\mathcal{O}(r))+c_2' r^{l+2}(1+\mathcal{O}(r))]$, respectively. For $-\frac{1}{2} \leqslant l < \frac{1}{2}$, which corresponds† to the problematic range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$ for V_-, both solutions are square integrable. However, in the above range $[-\frac{1}{2} \leqslant l < \frac{1}{2}]$, the eigenvalue problem for V_+ is well defined and one can determine a unique wavefunction $\psi^{(+)}$. A proper wavefunction (and from it the eigenvalues) for V_- can then be obtained by applying operator $A^\dagger$ on the solution of V_+ as shown in equation (8). In what follows, we will use two examples to explicitly describe the working of our approach.

Example (a): The Pöschl–Teller potential. Let us consider the Pöschl–Teller II superpotential

$$W(r) = A \tanh r - B \coth r \qquad (0 \leqslant r < \infty). \tag{9}$$

For $A > B$, the above superpotential corresponds to a case of unbroken SUSY. The corresponding supersymmetric partner potentials are given by

$$\begin{aligned} V_-(r) &= -A(A+1)\,\mathrm{sech}^2 r + B(B-1)\,\mathrm{cosech}^2 r + (A-B)^2 \\ V_+(r) &= -A(A-1)\,\mathrm{sech}^2 r + B(B+1)\,\mathrm{cosech}^2 r + (A-B)^2. \end{aligned} \tag{10}$$

Without loss of generality‡, we will assume $-\frac{1}{2} \leqslant A < \infty$ and $\frac{1}{2} \leqslant B < \infty$. To clearly see the ambiguity in the eigenvalue problem, we proceed with the analysis of the Schrödinger equation. The time-independent Schrödinger equation for $V_-(r)$ is given by

$$\frac{\mathrm{d}^2\psi^{(-)}}{\mathrm{d}r^2} + \left[E + \frac{A(A+1)}{\cosh^2 r} - \frac{B(B-1)}{\sinh^2 r} - (A-B)^2\right]\psi^{(-)}(r) = 0. \tag{11}$$

With a change of variables $y = -\sinh^2 r$ and $\psi(y) = (1-y)^{-(1/2)A}(y)^{(1/2)B}v(y)$, equation (11) can be cast in the form of a hypergeometric equation, i.e.

$$y(1-y)v'' + [(\tfrac{1}{2}+B) - (1-A+B)y]v' - \tfrac{1}{4}Ev = 0. \tag{12}$$

† The range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$ also corresponds to $-\frac{3}{2} < l \leqslant -\frac{1}{2}$; however, one can show that they also have exactly the same solutions for $\psi^{(-)}(r)$. Thus, by just limiting ourselves to $-\frac{1}{2} \leqslant l < \frac{1}{2}$, we can generate all solutions for the range $-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$.

‡ Since the potential $V_-(r)$ is a function of $A(A+1)$ and $B(B-1)$, it has same value for $A = -\frac{1}{2} \pm \zeta$ and $B = \frac{1}{2} \pm \zeta$ for an arbitrary real number ζ. We are choosing $A = -\frac{1}{2} + \zeta$ and $B = \frac{1}{2} + \zeta$.

The general solution is

$$\psi^{(-)}(r) = \cosh^{-A} r[c_1 \sinh^{B} r F(a', b', c'; -\sinh^2 r) + c_2 \sinh^{(1-B)} r F(a' + 1 - c', b' + 1 - c', 2 - c'; -\sinh^2 r)] \tag{13}$$

where the constants a', b', and c' are given by

$$a' = \tfrac{1}{2}(B - A + Q') \qquad b' = \tfrac{1}{2}(B - A - Q') \qquad c' = B + \tfrac{1}{2} \qquad \text{and} \qquad Q' = \sqrt{(B-A)^2 - E}. \tag{14}$$

Near the point $r \sim 0$, the solution reduces to

$$\psi^{(-)}(r) \overset{r\sim 0}{\to} [c_1 r^{B}(1 + \mathcal{O}(r)) + c_2 r^{(1-B)}(1 + \mathcal{O}(r))]. \tag{15}$$

Normalizability requires that $\psi^{(-)}$ be less singular than $1/\sqrt{r}$ near the origin. Thus, for $B \geqslant \frac{3}{2}$, the wavefunction becomes non-normalizable unless $c_2 = 0$. With $c_2 = 0$, a subsequent constraint coming from the requirement of the vanishing of the wavefunction at infinity (which is demanded by the normalizability) suffices to determine the eigenvalue E in terms of the parameters A and B. However, if $\frac{1}{2} \leqslant B < \frac{3}{2}$, both terms of the wavefunction for the Hamiltonian H_- are well defined near $r \sim 0$ (see (15))† and, hence, no constraints are placed on their coefficients from requiring normalizability. In such cases, we solve the eigenvalue problem for V_+ instead. The wavefunction for the Hamiltonian H_+ near the origin is given by

$$\psi^{(+)}(r) \overset{r\sim 0}{\to} [\tilde{c}_1 r^{(B+1)}(1 + \mathcal{O}(r)) + \tilde{c}_2 r^{(-B)}(1 + \mathcal{O}(r))]. \tag{16}$$

Clearly, normalizability of $\psi^{(+)}$ for $B < \frac{3}{2}$ requires that we set $\tilde{c}_2 = 0$. To determine the eigenvalues of H_+, we have to study the behaviour at infinity and, for that, one uses an alternative asymptotic form of the hypergeometric function

$$F(a, b, c; z) \overset{z\sim\infty}{\to} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}. \tag{17}$$

This leads to

$$\psi^{(+)}(r) \overset{r\sim\infty}{\to} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} e^{-Qr} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} e^{Qr} \tag{18}$$

where the constants a, b and c are given by

$$a = \tfrac{1}{2}(B - A + 2 + Q) \qquad b = \tfrac{1}{2}(B - A + 2 - Q) \qquad c = B + \tfrac{3}{2} \qquad \text{and} \qquad Q = \sqrt{(B-A+2)^2 - \bar{E}}. \tag{19}$$

† If B is exactly equal to $\frac{1}{2}$, then $\psi^{(-)}(r) \overset{r\sim 0}{\to} r^{1/2}[c_1(1 + \mathcal{O}(r)) + c_2 \log r(1 + \mathcal{O}(r))]$. In this case both terms are normalizable in the vicinity of the origin and, hence, eigenvalues cannot be determined unambiguously in the absence of further assumption.

Here, $\bar{E} = E + (A - B - 2)^2 - (A - B)^2$. The second term on the right-hand side of equation (18) must vanish to have a well-defined bound state†. This can be achieved if either a or $(c - b)$ is equal to a negative number (say $-k$). If $a = -k$, then the eigenvalues are given by

$$E_k^{(-)} = (A - B)^2 - [A - B - 2k - 2]^2 \qquad k = 0, 1, \ldots, n.$$

The integer n gives the number of bound states that the potential will hold and is related to the parameters A and B. It is the largest integer satisfying $Q > 0$, i.e. $A - B - 2 > 2n$. This condition also ensures that $\exp[-\int^r W(r')\,dr']$ is a well-defined function and, hence, one has a supersymmetric situation. Eigenvalues for Hamiltonian H_- will be the same as those for H_+ except that H_- will have an additional state (ground state) with zero energy. The eigenfunctions of H_+ are given by [4]

$$\psi^{(+)}(r) = (\sinh r)^{(1+B)}(\cosh r)^{-(A-1)} P_k^{(B+(1/2),-A+(1/2))}(\cosh 2r).$$

The eigenfunctions of the Hamiltonian H_- will be given by applying the operator $A^\dagger$ (defined in equation (1)) on the function $\psi^{(+)}$. Near the origin, $\psi^{(+)}$ is given by

$$\psi^{(+)}(r) \overset{r\sim 0}{\to} r^{B+1}.$$

Now operating $A^\dagger$ on $\psi^{(+)}$ lowers the power of r by unity and, hence,

$$\psi^{(-)}(r) \overset{r\sim 0}{\to} r^{B}.$$

Comparing this expression with equation (15), we see that SUSYQM automatically chooses the term with higher power of r, which is consistent with the prescription of [1]. Hence, the eigenvalues obtained will also be the same. Thus, this method provides a justification for the usual practice of dropping the term with lower power of r in case of ambiguity.

Instead, if the second condition holds, i.e. $c - b = -k$ then the eigenvalues are given by $E_k^{(-)} = (A - B)^2 - [A + B + 2k + 1]^2$, $k = 0, 1, \ldots, n$. The condition on A, for n-bound states in the second case, obtained by requiring that $Q > 0$, is given by $A < -B - 2n - 1$ which cannot be satisfied for any n as we have assumed $-\frac{1}{2} \leqslant A < \infty$ and $\frac{1}{2} \leqslant B < \infty$.

Example (b): Anyons in a spherically-symmetric potential. Here we consider a system of two anyons and proceed along similar lines as those above. The motion can be divided into centre-of-mass motion and the dynamics of the relative coordinate [2, 5]. The Schrödinger equation for the relative coordinate is then given by [5]

$$-\left[\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{1}{r^2}\left(m + \frac{\theta}{\pi}\right)^2 \psi\right] + (V(r) - E)\psi = 0 \tag{20}$$

where θ is the statistics parameter and $2\mu = 1$. Substituting $\psi = (\phi/\sqrt{r})$, we get

$$-\phi'' + [(\nu^2 - \tfrac{1}{4})/r^2 + (V(r) - E)]\phi = 0 \tag{21}$$

† We are assuming $Q > 0$, otherwise the first term in equation (18) will have to vanish instead. In either case, we will have the same answer since $F(a, b, c; z) = F(b, a, c; z)$.

where $\nu = (m+(\theta/\pi))$. Equation (21) can now be interpreted as a one-dimensional equation where the domain of the variable r is given by $0 \leqslant r < \infty$. Now, the important question is the boundary condition. This has been recently analysed by Roy and Tarrach [5] who conclude that more general boundary conditions $\phi'(0) = c\phi(0)$, $c \neq 0, \infty$ are forbidden because they break supersymmetry. It is interesting to observe that in the three-dimensional monopole problem, the above mentioned boundary conditions are also extremely important [6]. If we ask for the overlap of the two particles to be zero, we require $\psi \to 0$ as $r \to 0$. This is equivalent to saying that the configuration space has been reduced to $R^2 \times (R^2 - \{0\})/Z_2$. For ϕ, that would imply that it goes to zero faster than $\sqrt{r}$.

However, if we only stipulate a need of square integrability, it implies that ϕ be less singular than $1/\sqrt{r}$. The solutions of equation (21) are then of the form r^{l+1} and r^{-l} where l is given by $l(l+1) = (\nu^2 - \frac{1}{4})$. One can show that for $l \geqslant \frac{1}{2}$ or $l < -\frac{1}{2}$, only one of the above two solutions is square integrable near the origin and this leads to the unambiguous determination of eigenvalues and eigenfunctions. However, if $-\frac{1}{2} \leqslant l < \frac{1}{2}$ then both solutions vanish at the origin and are also square integrable. Hence, it is not possible to choose one over the other. For the anyon problem, one starts with a superpotential of the form [5] $W(r) = ((l+1)/r) + f(r)$ where $f(r)$ has to be suitably chosen to give the required spherically-symmetric potential. One can easily see that in the partner Hamiltonian, the singular term is of the form $(l+1)(l+2)/r^2$ and, hence, just like the Pöschl–Teller case, there will be no ambiguity in this sector. As has been worked out in the previous example, the application of the degeneracy theorem will then give the less singular wavefunction in the H_- sector.

Thus, we find that the SUSYQM-based formalism gives a clear-cut way of finding the eigenvalues and eigenfunctions of transition potentials in the region of ambiguity. Also, the eigenfunctions turn out to be of the same 'less singular' type that is commonly chosen in the literature [1].

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